DEGREE STRUCTURES OF FUNCTION SPACES OVER UNUSUAL GROUND TYPES

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ABSTRACT. In this article, we investigate the degree structures of various functions spaces and hyperspaces. One of our goals is to gain a topological understanding of higher-type computability using negative information, and to this end, we explore the degree structures of function spaces whose ground types are endowed with cofinite topology or its relatives.

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Date: November 13, 2024.

1. Introduction

1.1. **Summary.** A central object of study in computability theory is the degrees of non-computability (e.g. the Turing degrees and the enumeration degrees). Recently, a unified topological treatment of these notions of degrees has been proposed in [23, 22]; for example, the Turing degrees are the degrees of non-computability of points in Cantor space 2^ω (or Baire space ω^ω), the enumeration degrees are the degrees of non-computability of points in Scott domain $\mathcal{P}(\omega)$ (or equivalently the ω-power $\mathbb{S}^\omega$ of Sierpiński space), the continuous degrees are the degrees of points in Hilbert cube $[0,1]^\omega$, the cototal degrees are the degrees of points in maximal antichain space, and so on. In this way, several notions of degrees are unified as the degrees of non-computability of points in topological spaces [22]; see also [21, 23, 17]. Based on this unification theory, this article analyzes the degree structures of various (higher-type) function spaces.

In modern theory, the notion of *admissibly represented space* [37] is widely used as a class of topological spaces in which the notion of computability makes sense. The category of admissibly represented spaces is cartesian closed, which implies that one can discuss the degrees of non-computability of higher-type functions. In the classical theory of higher-type computability [33, 25], the higher-type function spaces with ω (or $\mathbb{R}$ or similar) as a ground type have been studied as a central subject, but it seems that, for example, higher-type function spaces with non-Hausdorff or non-sober ground types have rarely been studied.

Let us explain why it is worth considering non-Hausdorff/non-sober ground types. The key notion is computability using *negative data*. Here, an element of a set A and its complement is considered as positive and negative data about A, respectively. For example, the idea of computational learning using negative data has been around for a long time (see e.g. [18]), and in recent years, computable enumeration using only negative data has also been found to play an important role in the computability-theoretic study of symbolic dynamics, word problems of groups, and so on [19, 28, 1]. Topologically, enumerating a negative data of a set can be interpreted as computing that set under the *cofinite topology* (or its analogue), which is a typical non-Hausdorff T_1-space. Moreover, this kind of space is not sober, so it has not been a subject of much attention in most fields.

One abstract formulation of higher-type computability using negative data is to consider the *de Groot dual* (for example, the de Groot dual of the discrete topology on ω is the cofinite topology), and it seems that the computability-theoretic study of the de Groot dual is worth studying also in the context of synthetic topology; see [24]. We discuss this formulation, but also discuss other notions as well.

Our article focuses largely on the analysis of computability-theoretic degree structures on higher-type function spaces based on negative information (for example, higher-type function spaces with a space endowed with the cofinite topology as a ground type), which have been overlooked by classical theory.

In Section 2, we analyze the degree structures of hyperspaces $\mathcal{O}(X)$ of open sets in various spaces X; that is, function spaces whose codomain is the Sierpiński space. In Section 3, we analyze the degree structures of function spaces $C(X,Y)$ with more general codomains. In Section 4, we analyze the degrees of Π^0_1 singletons and the degree structures of the spaces of co-singletons (i.e., the de Groot duals) in function spaces. In Section 5, we perform a technical analysis on the complexity of higher-order function spaces. In Section 6, we introduce and analyze a notion of reducibility for points in function spaces by linear realizability.

1.2. **Preliminaries.** For the basics of computability theory, we refer the reader to Cooper [7] and Soare [42]. For higher order computability, see [34, 25]. For computable topology, see [4]. The notions we mainly use are also summarized in our previous article [22].

1.2.1. *Enumeration and Medvedev reducibility.* We review the definition of enumeration reducibility (see also Odifreddi [35, Chapter XIV], Cooper [6] & [7, Chapter 11]). Let $(D_e)_{e\in\omega}$ be a computable enumeration of all finite subsets of ω. Given $A, B \subseteq \omega$, we say that A *is enumeration reducible to* B (written $A \leq_e B$) if there is a c.e. set Φ such that

$$n \in A \iff (\exists e)\ [\langle n, e\rangle \in \Phi \text{ and } D_e \subseteq B].$$

The Φ in the above definition is called an *enumeration operator*. An enumeration operator induces a computable function on ω^ω, and indeed, $A \leq_e B$ iff there is a computable function $f : \omega^\omega \to \omega^\omega$ such that given an enumeration p of A, $f(p)$ returns an enumeration of B, where we say that $p \in \omega^\omega$ is an enumeration of A if $A = \{p(n) - 1 : p(n) > 0\}$ ($p(n) = 0$ indicates that we enumerate nothing at the n-th step).

Each equivalence class under the e-equivalence $\equiv_e := (\leq_e \cap \geq_e)$ is called an *enumeration degree* or simply *e-degree*. The e-degree of a set $A \subseteq \omega$ is written as $\deg_e(A)$. The e-degree structure forms an upper semilattice, where the join is given by the disjoint union $A \oplus B = \{2n : n \in A\} \cup \{2n+1 : n \in B\}$. We use the symbol $\mathcal{D}_e$ to denote the set of all e-degrees.

For $P, Q \subseteq \omega^\omega$, we say that P *is Medvedev reducible to* Q (written $P \leq_M Q$, [29]) if there is a partial computable function $\Psi :\subseteq \omega^\omega \to \omega^\omega$ such that for any $q \in Q$, $\Psi(q) \in P$. There is a natural embedding of the enumeration degrees into the Medvedev degrees of the Baire space, by taking a set A to the class of all enumerations of A.

1.2.2. *Represented spaces.* The central objects of study in modern computability theory are the represented spaces, which allow us to make sense of computability for most space of interest in everyday mathematics.

Definition 1.1. A *represented space* is a set X together with a partial surjection $\delta :\subseteq \omega^\omega \to X$. We often write $\mathcal{X}$ for a represented space.

We say that $p \in \omega^\omega$ is a *δ-name of x* if $x = \delta(p)$. We use $\mathtt{Name}_\delta(x)$ to denote the set of all δ-names of x, or just write $\mathtt{Name}(x)$, if the space is clear from the context. Hereafter, by a *point*, we mean a pair of a point $x \in X$ and the underlying represented space $\mathcal{X} = (X, \delta)$, denoted by $x\colon \mathcal{X}$ or $x\colon \delta$ or simply x.

For points x, y, we write $x \leq_T y$ iff $\mathtt{Name}(x)$ is Medvedev reducible to $\mathtt{Name}(y)$. Then the degree $\deg(x)$ of a point x is defined as the Medvedev degree of $\mathtt{Name}(x)$ [23, 22]. In particular, a point is computable iff it has a computable name. The degree of a point in a represented spaces X is called an *X-degree*.

A partial function $F :\subseteq \omega^\omega \to \omega^\omega$ is called a *realizer* of a partial function $f :\subseteq \mathcal{X} \to \mathcal{Y}$, if $\delta_{\mathcal{Y}}(F(p)) = f(\delta_{\mathcal{X}}(p))$ for any $p \in \mathrm{dom}(f\delta_{\mathcal{X}})$; that is, given a name p of x, $F(p)$ returns a name of $f(x)$. We then say that f is *computable* (respectively *continuous*), if f has a computable (respectively continuous) realizer.

1.2.3. *Second-countable spaces.* A *represented cb space* is a pair $(\mathcal{X},\beta)$ of a second-countable space $\mathcal{X}$ and an enumeration $\beta=(\beta_e)_{e\in\omega}$ of a countable open subbasis of $\mathcal{X}$. Here, "cb" stands for "countably based". If a represented cb space is T_0, then it is also called a represented cb$_0$ space. The enumeration β is called a *cb representation of* $\mathcal{X}$.

One of the key observations is that specifying a cb representation β of a second-countable T_0 space $\mathcal{X}$ is the same thing as specifying an embedding of $\mathcal{X}$ into the power set $\mathcal{P}\omega$ of ω endowed with the Scott topology (that is, basic open sets are $\{X\subseteq\omega : D\subseteq X\}$ where D ranges over finite subsets of ω). Hence, a cb$_0$ representation β (and the induced embedding) determines how a point $x\in\mathcal{X}$ is identified with a subset of the natural numbers. This observation entails the known fact that the Scott domain $\mathcal{P}\omega$ is a universal second-countable T_0 space, that is, every second-countable T_0 space embeds into $\mathcal{P}\omega$. We describe how an embedding $:\mathcal{X}\hookrightarrow\mathcal{P}\omega$ is induced from a cb representation β. One can identify a point x in a represented cb$_0$ space $(\mathcal{X},\beta)$ with the coded neighborhood filter

$$\mathtt{Nbase}_\beta(x)=\{e\in\omega : x\in\beta_e\}.$$

It is not hard to see that $\mathtt{Nbase}_\beta\colon\mathcal{X}\hookrightarrow\mathcal{P}\omega$ is a topological embedding. An enumeration of $\mathtt{Nbase}_\beta(x)$ is called a *β-name of* x, that is, for a $p:\omega\to\omega$,

$$p \text{ is a } \beta\text{-name of } x \iff \mathrm{rng}(p)=\mathtt{Nbase}_\beta(x),$$

where one can assume that the zeroth term β_0 is the whole space $\mathcal{X}$ without loss of generality. This allows us to perform the action of not enumerating anything. If β is clear from the context, we also use the symbol $\mathtt{Nbase}(x)$ instead of $\mathtt{Nbase}_\beta(x)$.

Clearly, a cb-representation β always induces a representation δ_β (in the sense of Definition 1.1) defined by $\delta_\beta(p)=x$ iff p is a β-name of x (i.e., p enumerates $\mathtt{Nbase}_\beta(x)$). This entails that $\mathtt{Nbase}_{\mathcal{X}}(x)$ is c.e. iff $x\colon\mathcal{X}$ is computable. In situations where no confusion is expected, we may speak of a cb representation and its induced representation interchangeably. We can also express computability of partial functions between represented cb spaces equivalently as a special case of computability on represented spaces, or in the language of enumeration reducibility: Saying that $f:\subseteq\mathcal{X}\to\mathcal{Y}$ is computable is equivalent to saying that there is a single enumeration operator Ψ such that

$$(\forall x\in\mathrm{dom}(f))\ [\mathtt{Nbase}(f(x))\leq_e\mathtt{Nbase}(x)\text{ via }\Psi].$$

To be more explicit:

$$f(x)\in\beta^Y_n \iff (\exists e)\ [\langle n,e\rangle\in\Psi\text{ and }(\forall i\in D_e)\ x\in\beta^X_i] \tag{1}$$

for any $x\in\mathrm{dom}(f)$, where β^X and β^Y are fixed countable bases of $\mathcal{X}$ and $\mathcal{Y}$, respectively.

1.2.4. *Function space representation.* As seen above, a second countable T_0-space can be treated as a represented space, but the latter is a much larger class. One of the important differences is the existence of an exponential (i.e., a function space). To be more precise, the category of second countable T_0-spaces is not cartesian closed, but the category of represented spaces is cartesian closed.

If X and Y are both cb$_0$-spaces, a continuous function $f\colon X\to Y$ is determined by a set $\Psi\subseteq\omega^2$. Thus, an enumeration of elements of Ψ can be thought of as a name of f, which gives a representation of the function space $C(X,Y)$.

Analyzing the formula (1), we observe that each element $\langle n, e\rangle \in \Psi$ determines the basic neighborhood

$$[e, n] := \left\{f \in C(X, Y) : f\left[\beta^X_{D_e}\right] \subseteq \beta^Y_n\right\},$$

where $\beta^X_{D_e} = \bigcap_{i \in D_e} \beta^X_i$. The important point is that this basic neighborhood is not necessarily open. In other words, the collection $([e, n])_{e,n\in\omega}$ gives a certain kind of countable basic neighborhood system on $C(X, Y)$, but does not give a countable open basis. This suggests that the function space $C(X, Y)$ is not necessarily second-countable.

A useful tool for explaining this is Arhangel'skii's notion of a network in general topology. A *network* for a topological space X is a collection $\mathcal{N}$ of subsets of X such that for any open set $U \subseteq X$ and a point $x \in U$ there exists $N \in \mathcal{N}$ with $x \in N \subseteq U$.

For example, the collection of basic neighborhoods $[e, n]$ gives a countable network for $C(X, Y)$. In fact, this collection is even better: It forms a countable *cs*-network (see e.g. [22]), but we will not use this notion, so we omit the explanation. The notion of a network behaves very well for function space construction, so it is deeply studied in the context of functional space topologies. Indeed, if X and Y are topological spaces having countable cs-networks (N^X_i) and (N^Y_j), respectively, the basic neighborhoods

$$[e, n] := \left\{f \in C(X, Y) : f\left[N^X_{D_e}\right] \subseteq N^Y_n\right\},$$

form a countable cs-network for the function space $C(X, Y)$, where $N^X_{D_e} = \bigcap_{i\in D_e} N^X_i$.

A T_0-space with a countable network can always be treated as a represented space. If $x \in N \in \mathcal{N}$ then N is called an $\mathcal{N}$-neighborhood of x. A local network at x is a collection $\mathcal{M}$ of $\mathcal{N}$-neighborhoods of x such that for any open set $U \subseteq \mathcal{X}$ with $x \in U$ there exists $N \in \mathcal{M}$ with $x \in N \subseteq U$. A countable network $\mathcal{N}$ for $\mathcal{X}$ induces the following representation of $\mathcal{X}$:

$$\delta_{\mathcal{N}}(p) = x \iff \{N_{p(n)} : n \in \omega\} \text{ is a local network at } x.$$

Roughly speaking, a name of x is an enumeration of a local network at x. We call $\delta_{\mathcal{N}}$ the *induced representation of* $\mathcal{X}$ (obtained from $\mathcal{N}$). We also use the symbol $\mathtt{Name}_{\mathcal{N}}(x)$ to denote the set of all enumerations of a local network at x, that is,

$$\mathtt{Name}_{\mathcal{N}}(x) = \delta^{-1}_{\mathcal{N}}\{x\} = \{p \in \omega^\omega : \delta_{\mathcal{N}}(p) = x\}$$

If $\mathcal{N}$ is clear from the context, we also use $\mathtt{Name}(x)$ instead of $\mathtt{Name}_{\mathcal{N}}(x)$.

1.2.5. *Hyperspace representation.* As a special case of the function space $C(X, Y)$, the case where Y is the *Sierpiński space* is particularly important. Here, the Sierpiński space $\mathbb{S}$ is the nontrivial connected two-point space; that is, the underlying set is $\{\top, \bot\}$ and the open sets are $\emptyset$, $\{\top\}$ and $\{\top, \bot\}$. Clearly, a set $A \subseteq X$ is open iff the characteristic function $\chi_A : X \to \mathbb{S}$ is continuous. Hence, we can always think of $C(X, \mathbb{S})$ as the hyperspace of open subsets of X. We use $\mathcal{O}(X)$ to denote $C(X, \mathbb{S})$ when we consider a point in $C(X, \mathbb{S})$ as an open subset of X. An open set A in X is *computable* if $\chi_A : X \to \mathbb{S}$ is computable.

A basic neighborhood in $\mathcal{O}(X)$ is of the form $[e] = \{U \in \mathcal{O}(X) : N^X_{D_e} \subseteq U\}$, which yields a countable cs-network for $\mathcal{O}(X)$. Note that the basic neighborhoods in $\mathcal{O}(X)$ are exactly the basic neighborhoods in $C(X, \mathbb{S})$ in the sense of Section 1.2.4. To see this, note that $N^X_{D_e} \subseteq U$ iff $\chi_U[N^X_{D_e}] \subseteq \{\top\}$.

1.2.6. *Admissible representation.* For a topological space $\mathcal{X} = (X, \tau)$, we say (following Schröder [39]) that $\delta\colon \subseteq \omega^\omega \to \mathcal{X}$ is *admissible* if it is continuous, and for any continuous function $\gamma\colon \subseteq \omega^\omega \to \mathcal{X}$ there exists a continuous function $\theta\colon \subseteq \omega^\omega \to \omega$ such that $\gamma = \delta \circ \theta$. Note that admissible representations are the ones which realize the coarsest quotient topology refining τ.

For example, the representation δ_β induced from a cb-representation is always admissible. In fact, if $\mathcal{N} = (N_e)_{e\in\omega}$ is a countable cs-network for a T_0-space X, Schröder [39] showed that the induced representation $\delta_\mathcal{N}$ always gives an admissible representation of X. Conversely, if $\delta\colon \subseteq \omega^\omega \to X$ is an admissible representation, then $(\delta[\sigma])_{\sigma\in\omega^{<\omega}}$ yields a countable cs-network. In this way, admissibly represented spaces correspond to T_0-spaces having countable cs-networks.

1.2.7. *Computable topology.* A *computable homeomorphism* $f\colon X \to Y$ is a computable bijection whose inverse function $f^{-1}\colon Y \to X$ is also computable. A *computable embedding* $f\colon X \to Y$ is a computable homeomorphism $f\colon X \to Z$ for some subspace $Z \subseteq Y$. A represented space X is *computably compact* if $cov\colon \mathcal{O}(X) \to \mathbb{S}$ is computable, where $cov(U) = \top$ iff $X = U$. If X is a represented cb_0 space, then X is computably compact iff it is compact and there is a computable enumeration of all finite open covers; that is, $\{e \in \omega : X = \bigcup_{n\in D} \beta_n\}$ is c.e., where $(\beta_n)_{n\in\omega}$ is a countable basis for X; see e.g. [36, 11].

1.2.8. *Degree Theory.* For the basics in degree theory in represented topological spaces, we refer the reader to Kihara-Ng-Pauly [22].

Recall that the degree of a point in a represented space X is called an X-degree. The ω^ω-degrees are called the *total* degrees, which may be identified with the Turing degrees in the classical sense. The $C([0,1],\mathbb{R})$-degrees are called the *continuous* degrees [31]. Note that the 2^ω-degrees are exactly the ω^ω-degrees, and the $\mathbb{R}^\omega$-degrees are exactly the $C([0,1],\mathbb{R})$-degrees. The $\mathcal{O}(\omega)$-degrees can be identified with the enumeration degrees, see e.g. [23, 22], so we often use an e-degree to mean an $\mathcal{O}(\omega)$-degree.

For a class Γ, a point x is *quasi-minimal with respect to* Γ*-degrees* or Γ*-quasi-minimal* if x is not computable, and for any $y \in \Gamma$ if $y \leq_T x$ then y is computable; see also [22]. We simply say that a point x is quasi-minimal if x is quasi-minimal with respect to total degrees (i.e., ω^ω-quasi-minimal).

1.3. **Basic observations.** The notions we have discussed so far are rather abstract, so they may be a little difficult to understand, so let us describe them concretely in a specific situation. Assume that X is second-countable T_0-space, that is, it is endowed with the T_0-topology generated by a countable collection $\beta = (B_n)_{n\in\omega}$ of subsets of X. Hereafter, we use B_D to denote $\bigcap_{d\in D} B_d$. Given $f\colon X \to Y$, we say that N is a *(coded) local network at* f if

$$(\forall x)(\forall e)\ [f(x) \in B_e \ \leftrightarrow\ (\exists D \text{ finite})\ [x \in B_D \text{ and } \langle D, e\rangle \in N]].$$

Here a finite set $D \subseteq \omega$ is identified with its canonical index. In higher type computability theory, a coded local network has also been called an *associate* [33].

Remark. This is consistent with the definition of a local network that has already been introduced. Recall that $[D, e] = \{f \in C(X, Y) : f[B_D] \subseteq B_e\}$ gives a countable cs-network for the function space $C(X, Y)$. Then N is a coded local network at f iff $([D, e] : \langle D, e\rangle \in N)$ is a local network at f.

Remark. As a special case (i.e., $Y = \mathbb{S}$), a (coded) local network at $U \in \mathcal{O}(X)$ is a set of finite sets D such that $B_D \subseteq U$, and the union of all B_D is U.

Observation 1.2. *A function f is continuous iff there is a local network at f.*

Proof. If f is continuous, then $f^{-1}[B_e]$ is open, and by second countability, $f^{-1}[B_e]$ can be written as the union of countably many basic open sets $(B_{D^e_n})_{n\in\omega}$. For each e, enumerate such $\langle D^e_n, e\rangle$ into N. It is easy to see that N is a local network at f. Conversely, if N is local network, then $f^{-1}[B_e] = \bigcup_{\langle D,e\rangle\in N} B_D$, which is open. Therefore, f is continuous. □

One of our aims is to compare the degree structures of various function spaces. For this purpose, the following observation is also useful.

Observation 1.3. *If Y is computably embedded into Z, then $C(X,Y)$ is computably embedded into $C(X,Z)$ for any X. In particular, $C(X,Y)$-degrees are included in the $C(X,Z)$-degrees.*

In particular, if Y is a represented cb_0-space, then every $C(X,Y)$-degree has a $C(X,\mathbb{S}^\omega)$-degree.

Corollary 1.4. *If Y is a represented cb_0 space, then the $C(X,Y)$-degrees are included in the $\mathcal{O}(\omega\times X)$-degrees.*

Proof. As Y is computably embedded into $\mathbb{S}^\omega$, then by Observation 1.3, $C(X,Y)$ is computably embedded into $C(X,\mathbb{S}^\omega) \simeq C(X\times\omega,\mathbb{S}) \simeq \mathcal{O}(\omega\times X)$. □

Recall that a computable open set in X is a computable point in $\mathcal{O}(X)$.

Observation 1.5. *If $A \subseteq X$ is a computable open set in a second-countable T_0 space X, then the $\mathcal{O}(A)$-degrees are included in the $\mathcal{O}(X)$-degrees.*

Proof. Any open set of A is an open set of X, and for any open set U of X, $U\cap A$ is an open set of A. Thus, for any $S \subseteq A$, given a local network N at S in $\mathcal{O}(X)$, $\{U\cap A : U \in N\}$ is a local network at S in $\mathcal{O}(A)$. Conversely, given a local network N at S in $\mathcal{O}(A)$, if B^A_D is enumerated into N, enumerate all open sets of the form $B^A_D \cup B^X_E$. This yields a local network at S in $\mathcal{O}(X)$. Thus, the degree of S in $\mathcal{O}(A)$ is exactly the degree of S in $\mathcal{O}(X)$. □

We also mention some topological properties on $C(X,Y)$. Define $\delta_{X\to Y}(p) = f$ iff $p \in \mathtt{Name}(f)$. We assume that $C(X,Y)$ is endowed with the final topology with respect to $\delta_{X\to Y}$. A set is saturated if it is an intersection of open sets. For a saturated compact set K and an open set U, it is not hard to see that $\{f : f[K] \subseteq U\}$ is open. As every set in a T_1-space is saturated, every compact-open set is open. Indeed, Schröder [37, Proposition 2.4.18 (4) and Proposition 4.2.5 (4)] pointed out that $C(X,Y)$ is topologized by the sequentialization of the compact-open topology.

Remark. Assume that X is T_1. If Y is T_i for $i \in \{1,2\}$ then so is $C(X,Y)$. This is because every singleton is compact.

1.3.1. *Characterization of reducibility.* For a space X equipped with a countable network $\mathcal{N}_X = (N_e)_{e\in\omega}$, we use the symbol $\mathtt{Net}(x)$ to denote the set of all coded local networks at $x \in X$, and $\mathtt{Name}(x)$ to denote the set of all enumerations of coded local networks at x.

Recall that $x \leq_T y$ iff $\mathtt{Name}(x)$ is Medvedev reducible to $\mathtt{Name}(y)$. However, to show $x \leq_T y$ we will often construct an enumeration operator Φ satisfying the following statement:

$$(\exists\Phi)(\forall N_y \in \mathtt{Net}(y))(\exists N_x \in \mathtt{Net}(x))\ [N_x \leq_e N_y \text{ via } \Phi]. \tag{2}$$

Lemma 1.6. *$x \leq_T y$ iff (2) holds for some enumeration operator Φ.*

Proof. ($\Leftarrow$) Given $\sigma \in \omega^{<\omega}$, let $\Psi(\sigma)$ be a sequence consisting of n's such that $\langle n, D$ for some $D \subseteq rng(\sigma)$ is enumerated into Φ by stage $|\sigma|$. Then Ψ yields a Medvedev reduction witnessing $x \leq_T y$. To see this, given a name p of y, we have a local network $N_y = rng(p)$ at y. Hence, $\Psi(p)$ is an enumeration of $\Phi(N_y)$, which is a local network at x.

($\Rightarrow$) Let Ψ be a Medvedev reduction witnessing $x \leq_T y$. If $\Psi(\sigma) = \tau$ then put $\langle \tau(n), rng(\sigma)\rangle \in \Phi$ for each $n < |\tau|$. We show that Φ is the desired enumeration operator. Let N_y be a local network at y. If $\sigma \in \omega^{<\omega}$ and $rng(\sigma) = D \subseteq N_y$ then σ can be extended to a name of y, so $\Psi(\sigma)$ must be extendible to a name of x; hence $\Psi(\sigma)$ consists of $\mathcal{N}_X$-neighborhoods of x. Thus, if $\Phi(\sigma) = \tau$ then $\tau(n)$ is an $\mathcal{N}_X$-neighborhood of x, so $\Phi(N_y)$ consists of $\mathcal{N}_X$-neighborhoods of x. An enumeration p of N_y is a name of y, so $\Psi(p)$ enumerates a local network N_x at x, which implies $N_x \subseteq \Phi(N_y)$. Hence, $\Phi(N_y)$ is a set of $\mathcal{N}_X$-neighborhood of x including N_x. Then one can see that $\Phi(N_y)$ is also a local network at x. □

In order to compare degrees of points in second countable spaces and points in function spaces, we fix the natural embedding of e-degrees into the Medvedev degrees, by associating each set $A \subseteq \omega$ with $\mathtt{Enum}(A) = \{f \in \omega^\omega : rng(f) = A\}$.

The following lemma is useful:

Lemma 1.7. *A point x has an e-degree iff there is a "canonical" local network at x which is e-reducible to any local network at x via a single enumeration operator; that is,*

$$(\exists\Phi)(\exists E \in \mathtt{Net}(x))(\forall N \in \mathtt{Net}(x))\ [E \leq_e N \textit{ via } \Phi].$$

Proof. Assume that x has an e-degree. Then there is $A \subseteq \omega$ such that $\mathtt{Enum}(A) \equiv_M \mathtt{Name}(f)$. Let Φ witness $\leq_M$ and Ψ witness $\geq_M$. Given an enumeration of A, we can construct the set S of all finite enumerations of finite subsets of A. Then define E be the set of all neighborhoods of x enumerated by $\Psi(\sigma)$ for some $\sigma \in S$, that is, $E = \{\Psi(\sigma; n) : \sigma \in S,\ n < |\Psi(\sigma)|\}$. This is clearly a local network at x. □

Moreover, such a canonical local network E represents the e-degree of x. In many cases, a canonical (coded) local network is the maximum local network w.r.t. $(N_e)_{e\in\omega}$; that is, $E = \{e \in \omega : x \in N_e \in \mathcal{N}\}$. In higher type computability theory, the maximum local network has also been called the *principal associate* [33].

2. Hyperspace of open sets: $C(X, \mathbb{S})$

In this paper we will consider $C(X, Y)$ for various well-known combinations of second-countable T_0-spaces X and Y. The aim is to examine their degree structures. The first case we will consider is when Y is the Sierpiński space $\mathbb{S}$ (see Section 1.2.5); that is, we consider hyperspaces $\mathcal{O}(X) \simeq C(X, \mathbb{S})$ for various X. The results are summarized in Table 1.

Hyperspace	Degree structure	Reference
$\mathcal{O}(\omega_{\rm cof})$	the least Medvedev degree	Trivial
$\mathcal{O}(\mathbb{R}_<)$	exactly all $\mathbb{R}_<$-degrees	Proposition 2.3
$\mathcal{O}(2^\omega)$	exactly all e-degrees	Corollary 2.2
$\mathcal{O}([0,1]^\omega)$		Corollary 2.2
$\mathcal{O}(\omega^\omega_{\rm co})$		Proposition 2.4
$\mathcal{O}(\omega^\omega_{\rm cof})$		Proposition 2.7
$\mathcal{O}\left(\mathcal{A}_{\max}(\omega^{<\omega})\right)$		Corollary 2.9
$\mathcal{O}(\mathbb{S}^\omega)$		Corollary 2.11
$\mathcal{O}\left(\mathbb{Q}\right)$	strictly contains all e-degrees, and is strictly contained in the $\mathcal{O}(\omega^\omega)$-degrees	Theorem 2.22
$\mathcal{O}(\omega^\omega)$	strictly contains all e-degrees	[24]
$\mathcal{O}(\mathbb{R}^\omega)$	exactly all $\mathcal{O}(\omega^\omega)$-degrees	Corollary 2.15
$\mathcal{O}\left(C(\omega_{\rm cof})\right)$	exactly all $\mathcal{O}(\omega^\omega)$-degrees	Theorem 2.18
$\mathcal{O}\left(\mathtt{Name}(\omega\langle 2\rangle)\right)$	strictly contains all $\mathcal{O}(\omega^\omega)$-degrees	Theorem 2.29

TABLE 1. Degree structures of hyperspaces of open sets

2.1. **Second countable hyperspaces of open sets.** In general, even if X is second-countable, $O(X)$ is not necessarily second-countable. For example, $O(\omega^\omega)$ is not second-countable. Here, we look for the conditions under which $O(X)$ is second-countable.

2.1.1. *Hyperspaces on metric spaces.* A proper metric space is a metric space all of whose closed ball is compact. An *computable proper metric space* is a computable metric space (X,d,α) such that, given a rational closed ball B, one can enumerate all finite basic open covers of B. To be more precise, (X,d) is a proper metric space, $\alpha=(\alpha_i)_{i\in\omega}$ is a dense sequence of points in X, the map $(i,j)\mapsto d(\alpha_i,\alpha_j)$ is computable, and there exists a computable function Φ such that for any $i\in\omega$ and positive rational $p\in\mathbb{Q}$, $\Phi(i,p)$ enumerates all finite sets D with $\overline{B}(\alpha_i;p)\subseteq\bigcup_{\langle j,q\rangle\in D}B(\alpha_j;q)$, where $B(x;r)$ and $\overline{B}(x;r)$ are the open and closed balls of center x and radius r.

Note that a computable (proper) metric space X can be always thought of as a represented cb$_0$ space. Indeed, $(B(\alpha_i;p))_{i,p}$ gives a cb$_0$-representation of X.

Proposition 2.1. *If X is a computable proper metric space, then the $\mathcal{O}(X)$-degrees are included in the e-degrees.*

Proof. Given an open set U in X, $E=\{\langle i,q\rangle : B(\alpha_i;q)\subseteq U\}$ forms a coded local network at U. If N is a coded local network at U, then observe that $N'=\{\overline{B}(\alpha_i;q/2):\langle i,q\rangle\in N\}$ is also a local network at U. For $\langle i,q\rangle\in N$, as $\overline{B}(\alpha_i;q/2)\subseteq U$ is compact and N yields an open cover of U, one can effectively find a finite set $D\subseteq N$ such that $\overline{B}(\alpha_i;q/2)\subseteq\bigcup_{\langle j,p\rangle\in D}B(\alpha_j;p)$. Then enumerate all rational open balls included in $\bigcup_{\langle j,p\rangle\in D}B(\alpha_j;p)$. This procedure eventually enumerates all elements in E. By Lemma 1.7, this implies that U has an e-degree. □

In particular, if X is a computable compact metric space, then $\mathcal{O}(X)$ has an e-degree.

Corollary 2.2. *The $\mathcal{O}(2^\omega)$-degrees and the $\mathcal{O}([0,1]^\omega)$-degrees are exactly the e-degrees.*

Proof. By Proposition 2.1, these degrees are all e-degrees since 2^ω and $[0,1]^\omega$ are computably compact. Conversely, observe that ω can be embedded into 2^ω and $[0,1]^\omega$, where the embedded images are computably open. Thus, by Observations 1.3 and 1.5, any $\mathcal{O}(\omega)$-degree, i.e., any e-degree, is an $\mathcal{O}(2^\omega)$-degree and an $\mathcal{O}([0,1]^\omega)$-degree. □

2.1.2. *Hyperspaces on non-metrizable spaces.* Let $\mathbb{R}_<$ be the space of real numbers endowed with the lower topology; that is, each open set in $\mathbb{R}_<$ is of the form $(x,\infty) = \{y \in \mathbb{R} : y > x\}$ for some real x.

Proposition 2.3. *The $\mathcal{O}(\mathbb{R}_<)$-degrees are exactly the $\mathbb{R}_<$-degrees.*

Proof. An open set in $\mathbb{R}_<$ is simply an interval (x,∞) for some x. Thus, the collection $((q,\infty))_{q\in\mathbb{Q}}$ is a countable network for $\mathcal{O}(\mathbb{R}_<)$. Then a local network N at $(x,\infty) \in \mathcal{O}(\mathbb{R}_<)$ is a collection of sets of the form $\{U \in \mathcal{O}(\mathbb{R}_<) : (q,\infty) \subseteq U\}$, where the infimum of q's is exactly x. Now, $\mathtt{Name}((x,\infty))$ consists of all enumerations of a coded local network at (x,∞), which is in turn an arbitrary collection of rationals with infimum x. Given any member of $\mathtt{Name}((x,\infty))$, we can clearly enumerate the right cut of x, and hence the $\mathtt{Nbase}(x)$ in $\mathbb{R}_<$. The converse is also clear. □

Let $\omega^\omega_{\rm co}$ be the space of all sequences of natural numbers endowed with the cocylinder topology (see [22]); that is, a subbasis element is the complement $\omega^\omega \setminus [\sigma]$ of a cylinder $[\sigma] = \{f \in \omega^\omega : f \text{ extends } \sigma\}$ for some $\sigma \in \omega^{<\omega}$.

Proposition 2.4. *The $\mathcal{O}(\omega^\omega_{\rm co})$-degrees are exactly the e-degrees.*

Proof. By definition, a basis of $\omega^\omega_{\rm co}$ consists of sets of the form $\bigcap_{\sigma\in D}(\omega^\omega \setminus [\sigma])$ for a finite set $D \subseteq \omega^{<\omega}$. These sets are rewritten as $\omega^\omega \setminus [D]$, where $[D] = \bigcup_{\sigma\in D}[\sigma]$. Hence, a basic neighborhood in $\mathcal{O}(\omega^\omega_{\rm co})$ is of the form $\{U \in \mathcal{O}(\omega^\omega_{\rm co}) : \omega^\omega \setminus [D] \subseteq U\}$. This condition is equivalent to $\omega^\omega \setminus U \subseteq [D]$.

Given $U \in \mathcal{O}(\omega^\omega_{\rm co})$, consider $F = \omega^\omega \setminus U$. It is not hard to see that a name of U is a sequence $(D_n)_{n\in\omega}$ of sets of finite strings such that $F = \bigcap_{n\in\omega}[D_n]$. Note that $T_U = \{\sigma \in \omega^{<\omega} : F \cap [\sigma] \neq \emptyset\}$ is a full-or-finitely-branching tree; that is, for any $\sigma \in \omega^{<\omega}$, either all strings extending σ are contained in T_U or σ has only finitely many immediate successors in T_U. This is because if σ is not full-branching, then $\sigma^\frown i \notin T_U$ for some $i \in \omega$, which is witnessed by D_n for some n; that is, no initial segment of $\sigma^\frown i$ is contained in D_n. Since D_n is finite, at most finitely many immediate successors of σ can be extended to an element in D_n. This means that σ is finitely branching in T_U.

For each $\sigma \in \omega^{<\omega}$ consider ${\rm succ}_U(\sigma) = \{i \in \omega : \sigma^\frown i \in T_U\}$. This is either ω or finite. Now let $E = \{\langle\sigma, C\rangle : \sigma \in \omega^{<\omega} \text{ and } C \subseteq \omega \text{ finite such that } {\rm succ}_U(\sigma) \subseteq C\}$. Note that D depends only on U and not on any particular name of U. We claim that $\mathtt{Name}(U) \equiv_M \mathtt{Enum}(E)$, and hence U has an e-degree.

To see $\mathtt{Name}(U) \geq_M \mathtt{Enum}(E)$, given any name of U we can effectively enumerate the complement of T_U and in particular the complement of ${\rm succ}_U(\sigma)$. Moreover, one can compute a partial function $b_U :\subseteq \omega^{<\omega} \to \omega$ such that if ${\rm succ}_U(\sigma)$ is finite then $b_U(\sigma)$ is defined and $i \leq b_U(\sigma)$ for any $i \in {\rm succ}_U(\sigma)$. This is because if ${\rm succ}_U(\sigma)$ is finite then as in the above argument, there is n such that at most finitely many immediate successors of σ can be extended to an element in D_n. Since D_n is finite, one can compute $b_U(\sigma) := \max\{i : \sigma^\frown i$ extends to an element in $D_n\}$. Now, at stage s, we have the approximations $\mathrm{succ}_U(\sigma)[s]$ and $b_U(\sigma)[s]$, and then enumerate all finite sets C such that $\{i \leq b_U(\sigma)[s] : i \in \mathrm{succ}_U(\sigma)[s]\}$ if $b_U(\sigma)[s]$ is defined. This procedure enumerates all elements of E.

To see $\mathtt{Name}(U) \leq_M \mathtt{Enum}(E)$, let an enumeration $E = (\langle \sigma_s, C_s\rangle)_{s\in\omega}$ be given. Let D_0 be the singleton $\{\langle\rangle\}$ consisting of the empty string $\langle\rangle$. Then define $D_{s+1} = (D_s \setminus \{\sigma_s\}) \cup \{\sigma_s{}^\frown i : i \in C_s\}$ for any $s \in \omega$. It is not hard to see that $F = \bigcap_{n\in\omega}[D_n]$, so $D = (D_s)_{s\in\omega}$ is a name of U.

Conversely, let $A \subseteq \omega$ be given, and we wish to show that $\mathtt{Enum}(A)$ has an $\mathcal{O}(\omega^\omega_{\mathrm{co}})$-degree. Consider a closed set $F \subseteq \omega^\omega$ in Baire space such that $0^n1^\omega \in F$ iff $n \notin A$. Recall that an $\mathcal{O}(\omega^\omega_{\mathrm{co}})$-name of $\omega^\omega \setminus F$ is a collection of finite covers $[D_n]$ of F such that $F = \bigcap_{n\in\omega}[D_n]$. To see that $\omega^\omega \setminus F \in \mathcal{O}(\omega^\omega_{\mathrm{co}})$ is reducible to $A \in \mathcal{O}(\omega)$, if we see $n \in A$, we enumerate a finite cover of F which do not cover $[0^n1]$. To see that A is reducible to F, if we see that F has a finite cover which does not cover 0^n1^ω, then enumerate $n \in A$. Thus, $A \in \mathcal{O}(\omega)$ is T-equivalent to $\omega^\omega \setminus F \in \mathcal{O}(\omega^\omega_{\mathrm{co}})$. □

2.1.3. *Hyperspaces on spaces with compact bases.* Let us discuss a general condition for all points in X to have e-degrees. As we have already seen, points in a computable metric space X with computably compact balls have e-degrees. Therefore, one may expect that some compactness property for the basis elements plays an important role. Indeed, it is known that a second-countable Hausdorff space X is locally compact if and only if $\mathcal{O}(X)$ is second-countable, cf. de Brecht-Schröder-Selivanov [10]. Since we mainly consider non-Hausdorff spaces, we need a slightly different version, which requires a stronger condition than local compactness.

Lemma 2.5. *Let (X, β) be a represented cb_0 space whose basis $\beta = (B_e)_{e\in\omega}$ is uniformly computably compact; that is, given e, one can effectively enumerate all finite open covers of the compact open set B_e. Then the $\mathcal{O}(X)$-degrees are included in the e-degrees.*

Proof. Given an open set $\mathcal{U} \in \mathcal{O}(X)$, we show that $E = \{e : B_e \subseteq \mathcal{U}\}$ is a canonical local network at $\mathcal{U}$. To see this, note that uniform computable compactness is equivalent to computability of $cov\colon \omega \times \mathcal{O}(X) \to \mathbb{S}$ defined by $cov(e, \mathcal{U}) = \top$ iff $B_e \subseteq \mathcal{U}$. This implies that $E = \{e : cov(e, \mathcal{U}) = \top\}$ is uniformly c.e. relative to any name of $\mathcal{U}$; in particular, E is e-reducible to any $N \in \mathtt{Net}(\mathcal{U})$ via a single enumeration operator. Hence, E is canonical, so $\mathcal{U}$ has an e-degree by Lemma 1.7. □

There are surprisingly many spaces to which this lemma can be applied. Let us look at the first example. Let ω_{cof} be the space of natural numbers endowed with the cofinite topology; that is, cofinite subsets of ω form a basis. One can easily see that ω_{cof} is compact, so $\omega^\omega_{\mathrm{cof}}$ is also compact by Tychonoff's theorem. Degree theoretically, the space $\omega^\omega_{\mathrm{cof}}$ characterizes the graph-cototal degrees [1, 22].

We explicitly describe a basis of $\omega^\omega_{\mathrm{cof}}$. Observe that basic open sets $[i, D] = \{x \in \omega^\omega \mid x(i) \notin D\}$ form a subbasis. Thus, for a finite collection $(D_i)_{i\in I}$ of finite subsets of ω, $[(D_i)_{i\in I}] = \{x \in \omega^\omega : (\forall i \in I)\ x(i) \notin D_i\}$ is open, and such open sets form a basis, referred to as the standard basis.

Proposition 2.6. *The standard basis for $\omega^\omega_{\mathrm{cof}}$ is uniformly computably compact.*

Proof. If D is finite, $[D] = \{n \in \omega_{\mathrm{cof}} : n \notin D\}$ is homeomorphic to ω_{cof}. Thus, $[i, D] \simeq \omega_{\mathrm{cof}}^i \times [D] \times \omega_{\mathrm{cof}}^\omega$ is homeomorphic to ω_{cof}. Similarly, $[(D_i)_{i\in I}]$ is homeomorphic to $\omega_{\mathrm{cof}}^\omega$, which is compact.

It remains to show that one can effectively enumerate all finite open covers of basis elements $[(D_i)_{i\in I}]$. Indeed, we show that the covering relation $[(D_i)_{i\in I}] \subseteq \bigcup_{k<\ell}[(E_j^k)_{j\in J_k}]$ is decidable, where $(D_i)_{i\in I}$ and $(E_j^k)_{j\in J_k}$ for $k < \ell$ are finite collections of finite sets. To be explicit, $[(D_i)_{i\in I}] \subseteq \bigcup_{k<\ell}[(E_j^k)_{j\in J_k}]$ iff

$$\text{for any } \gamma \in \textstyle\prod_{k<\ell} J_k \text{ there exists } j \in \omega \text{ such that } \bigcap_{\gamma(k)=j} E_j^k \subseteq D_j \tag{3}$$

where we put $\bigcap \emptyset = \omega$ and $D_j = \emptyset$ for $j \notin I$. Since we only need to check finitely many γ's and $j \in \bigcup_{k<\ell} J_k$, the truth-value of (3) is computable in a finite number of steps.

To show the equivalence, note that $[(D_i)_{i\in I}] \subseteq \bigcup_{k<\ell}[(E_j^k)_{j\in J_k}]$ iff for any $x \in \omega^\omega$, $x(i) \notin D_i$ for any $i \in I$ implies the existence of $k < \ell$ such that $x(j) \notin E_j^k$ for any $j \in J_k$. Considering the contrapositive, for any $x \in \omega^\omega$

$$(\forall k < \ell)(\exists j \in J_k)\ x(j) \in E_j^k \implies (\exists i \in I)\ x(i) \in D_i. \tag{4}$$

For (3)⇒(4): assuming the premise of (4) we get a function $\gamma \in \prod_{k<\ell} J_k$ such that $x(\gamma(k)) \in E^\ell_{\gamma(k)}$ for any $k < \ell$. Thus, $x(j) \in \bigcap_{\gamma(k)=j} E_j^k$ for any j. By (3), we get j such that $x(j) \in \bigcap_{\gamma(k)=j} E_j^k \subseteq D_j$. In this case, we must have $D_j \neq \emptyset$, so $j \in I$.

For (4)⇒(3): We show the contrapositive. By the negation of (3), there exists $\gamma \in \prod_{k<\ell} J_k$ such that $\bigcap_{\gamma(k)=j} E_j^k \not\subseteq D_j$ for any $j \in \omega$. Then there exists $x \in \omega^\omega$ such that $x(j) \in \bigcap_{\gamma(k)=j} E_j^k$ but $x(j) \notin D_j$ for any j. Then, for any $k < \ell$, $x(j) \in E_j^k$ for $j = \gamma(k) \in J_k$, so the premise of (4) holds; however, $x(j) \notin D_j$ for any j, so (4) fails. □

Corollary 2.7. *The $\mathcal{O}(\omega_{\mathrm{cof}}^\omega)$-degrees are exactly the e-degrees.*

Proof. By Lemma 2.5 and Proposition 2.6.

It remains to show that every $\mathcal{O}(\omega)$-degree is an $\mathcal{O}(\omega_{\mathrm{cof}}^\omega)$-degree. To see this, note that $[n, m] = \{g \in \omega_{\mathrm{cof}}^\omega : g(n) \neq m\}$ is open in $\omega_{\mathrm{cof}}^\omega$. Thus, given $S \subseteq \omega$, consider the open set $U_S = \bigcup_{n\in S}[n, 0]$. Note that $\{\langle n, 0\rangle : n \in S\}$ is a local network at U_S, which is e-reducible to S. As already seen, U_S has a canonical local network, which is $E(U_S) = \{\langle a, b\rangle : [a, b] \subseteq U_S\}$ by Lemma 2.5. We now have $E(U_S) \leq_e S$, so it remains to show that $S \leq_e E(U_S)$. Then observe that $a \in S$ if and only if $\langle a, 0\rangle \in E(U_S)$. This is because, if $a \in S$, clearly $[a, 0] \subseteq \bigcup_{n\in S}[n, 0]$. Conversely, if $a \notin S$, then consider a function $g\colon \omega \to \omega$ defined by $g(a) = 1$ and $g(n) = 0$ for any $n \neq a$. Then, clearly $g \in [a, 0]$, but $g \notin [n, 0]$ for any $n \neq a$. Thus, $g \notin \bigcup_{n\in S}[n, 0]$ since $a \notin S$. Hence, $g \in [a, 0] \setminus U_S$, so $[a, 0] \not\subseteq U_S$. Hence, $\langle a, 0\rangle \notin E(U_S)$. This shows $S \leq_e E(U_S)$. □

There is an important class of enumeration degrees called *cototal* degrees [28, 1], which can be characterized topologically as the degrees of points in *computable G_δ* spaces; see [22]. In fact, all cototal degrees can be realized by points in a single computable G_δ space $\mathcal{A}_{\max}(\omega^{<\omega})$, the maximal antichain space [22]. A point in $\mathcal{A}_{\max}(\omega^{<\omega})$ is a maximal antichain in $\omega^{<\omega}$ with respect to the prefix relation, and a basic open set is of the form $[D] = \{A \in \mathcal{A}_{\max}(\omega^{<\omega}) : A \cap D = \emptyset\}$, where D is a finite subset of $\omega^{<\omega}$. The $[D]$'s form a basis for $\mathcal{A}_{\max}(\omega^{<\omega})$, referred to as the standard basis.

Proposition 2.8. *The standard basis for the maximal antichain space $\mathcal{A}_{\max}(\omega^{<\omega})$ is uniformly computably compact.*

Proof. We first show compactness of a basic open set $[E]$ in the maximal antichain space. Let $([D_n])_{n\in\omega}$ be an open cover of $[E]$. We define a finitely branching tree T as follows: $(\sigma_0, \sigma_1, \dots, \sigma_\ell) \in T$ iff $\sigma_n \in D_n \setminus E$ for all $n \le \ell$, and $\{\sigma_n\}_{n\le\ell}$ is an antichain. Suppose for the sake of contradiction that T has an infinite path $(\sigma_n)_{n\in\omega}$. Then, $\{\sigma_n\}_{n\in\omega}$ forms an antichain (allowing duplicates).

We claim that there exists a maximal antichain $A \subseteq \omega^{<\omega} \setminus E$ including $\{\sigma_n\}_{n\in\omega}$. To see this, first take any maximal antichain B including $\{\sigma_n\}_{n\in\omega}$. If $\sigma \in B \cap E$, then $B \setminus \{\sigma\} \cup \{\sigma^\frown\tau : \tau \in \omega^s\}$ is still a maximal antichain, where s is a number which is greater than the length of all strings in E. Repeating this process, we eventually obtain a maximal antichain A that does not intersect with E, where each σ_n remains in A since $\sigma_n \notin E$. This verifies the claim.

Now we get $A \in [E]$, but $A \cap [D_n] \neq \emptyset$ for all $n \in \omega$, and therefore, $A \notin \bigcup_n [D_n]$, which contradicts our assumption that $([D_n])$ is an open cover of $[E]$. Therefore, T has no infinite path, and by König's lemma, there is ℓ such that T has no nodes of length $\ge \ell$.

We claim that $([D_n])_{n<\ell}$ is a finite open cover of $[E]$. Otherwise, there is a maximal antichain $A \in [E]$ such that $A \notin \bigcup_{n<\ell}[D_n]$. That is, $A \cap D_n \neq \emptyset$ for all $n < \ell$. Choose $\sigma_n \in A \cap D_n$ for each $n < \ell$, and then $(\sigma_n)_{n<\ell} \notin T$ since T has no nodes of length $\ge \ell$. We also have $\sigma_n \in D_n \setminus E$ since $A \cap E = \emptyset$. By the definition of T, this means that the set $\{\sigma_n\}_{n<\ell}$ is not an antichain (when eliminating duplicates), so there are $i \neq j$ such that $\sigma_i \neq \sigma_j$ and σ_i is comparable with σ_j. However, $\{\sigma_i, \sigma_j\} \subseteq A$, which contradicts that A is an antichain. Consequently, $([D_n])_{n\le\ell}$ is a finite subcover of $([D_n])_{n\in\omega}$.

Consequently, $[E]$ is compact. Moreover, finiteness of a finitely branching tree is recognizable, so this verifies uniform computable compactness of basis elements $[E]$. Indeed, the above argument gives a decidable characterization for the covering relation $[E] \subseteq \bigcup_{n<\ell}[D_n]$ for finite sets $D_n, E \subseteq \omega^{<\omega}$. To be explicit, $[E] \not\subseteq \bigcup_{n<\ell}[D_n]$ iff there exists an antichain $\{\sigma_n\}_{n<\ell}$ such that $\sigma_n \in D_n \setminus E$ for each $n < \ell$. ☐

Corollary 2.9. *The $\mathcal{O}(\mathcal{A}_{\max}(\omega^{<\omega}))$-degrees are exactly the e-degrees.*

Proof. By Lemma 2.5 and Proposition 2.8.

It remains to show that every $\mathcal{O}(\omega)$-degree is an $\mathcal{O}(\mathcal{A}_{\max}(\omega^{<\omega}))$-degree. To see this, recall that $[D] = \{A \in \mathcal{A}_{\max}(\omega^{<\omega}) : A \cap D = \emptyset\}$ is open in $\mathcal{A}_{\max}(\omega^{<\omega})$. Thus, given $S \subseteq \omega$, consider the open set $U_S = \bigcup_{n\in S}[\tilde{n}]$, where $\tilde{n}$ is the singleton $\{\langle n\rangle\}$. Note that $\{\tilde{n} : n \in S\}$ is a local network at U_S, which is e-reducible to S. As already seen, U_S has a canonical local network, which is $E(U_S) = \{D : [D] \subseteq U_S\}$ by Lemma 2.5. We now have $E(U_S) \le_e S$, so it remains to show that $S \le_e E(U_S)$. Then observe that $a \in S$ if and only if $\tilde{a} \in E(U_S)$. This is because, if $a \in S$, clearly $[\tilde{a}] \subseteq \bigcup_{n\in S}[n, 0]$. Conversely, if $a \notin S$, then consider the set $A = \{\langle a, m\rangle : m \in \omega\} \cup \{\langle n\rangle : n \neq a\}$. Then A is clearly a maximal antichain in $\omega^{<\omega}$, and $A \in [\tilde{a}]$; however $A \notin [\tilde{n}]$ for any $n \neq a$. Thus, $A \notin \bigcup_{n\in S}[\tilde{n}]$ since $a \notin S$. Hence, $A \in [\tilde{a}] \setminus U_S$, so $[\tilde{a}] \not\subseteq U_S$. Hence, $\tilde{a} \notin E(U_S)$. This shows $S \le_e E(U_S)$. ☐

The product Sierpiński space $\mathbb{S}^\omega$ also has the similar property. For a finite set $D \subset \omega$, we use $[D]$ to denote the set of all A such that $D \subseteq A$. These form a basis of the topology on $\mathbb{S}^\omega$, which is referred to as the standard basis.

Proposition 2.10. *The standard basis for the universal second-countable T_0 space $\mathbb{S}^\omega$ is uniformly computably compact.*

Proof. An open set $\mathcal{U}$ in $\mathbb{S}^\omega$ is of the form $\bigcup_{E \in U}[E]$, where U is a collection of finite sets. For any such U, we claim that $[D] \subseteq \mathcal{U}$ if and only if there is $E \subseteq D$ such that $E \in U$. For the backward implication, $E \subseteq D$ implies $[D] \subseteq [E]$, so $[D] \subseteq \bigcup_{E \in U}[E] = \mathcal{U}$. For the forward implication, assume $[D] \subseteq \mathcal{U}$, which means that $D \subseteq A$ implies $E \subseteq A$ for some $E \in U$. Considering $A = D$, we get $E \subseteq D$ for some $E \in U$. This verifies the claim. This shows that, if $[D]$ is covered by a family $([E])_{E \in U}$, it is covered by a single element $[E]$; in particular, $[D]$ is compact. Moreover, the characterization of the covering relation $[D] \subseteq \mathcal{U}$ is effective, which verifies uniform computable compactness. □

Corollary 2.11. *The $\mathcal{O}(\mathbb{S}^\omega)$-degrees are exactly the e-degrees.*

Proof. By Lemma 2.5 and Proposition 2.10.

It remains to show that every $\mathcal{O}(\omega)$-degree is an $\mathcal{O}(\mathbb{S}^\omega)$-degree. To see this, recall that $[D] = \{A \subseteq \omega : D \subseteq A\}$ is open in $\mathbb{S}^\omega$. Thus, given $S \subseteq \omega$, consider the open set $U_S = \bigcup_{n \in S}[\{n\}]$. Note that $\{\{n\} : n \in S\}$ is a local network at U_S, which is e-reducible to S. As already seen, U_S has a canonical local network, which is $E(U_S) = \{D : [D] \subseteq U_S\}$ by Lemma 2.5. We now have $E(U_S) \leq_e S$, so it remains to show that $S \leq_e E(U_S)$. Then observe that $a \in S$ if and only if $\{a\} \in E(U_S)$. This is because, if $a \in S$, clearly $[\{a\}] \subseteq \bigcup_{n \in S}[n, 0]$. Conversely, if $a \notin S$, then $\{a\} \in [\{a\}]$ but $\{a\} \notin [\{n\}]$ for any $n \neq a$; hence, $\{a\} \notin \bigcup_{n \in S}[\{n\}]$ since $a \notin S$. Therefore, $A \in [\{a\}] \setminus U_S$, so $[\{a\}] \not\subseteq U_S$; thus $\{a\} \notin E(U_S)$. This shows $S \leq_e E(U_S)$. □

2.2. **Non-second countable hyperspaces of open sets.** We turn our attention to non-second-countable hyperspaces of open sets.

2.2.1. *The hyperspaces on quasi-Polish spaces.* Recall that $\mathcal{O}(\omega)$ is a universal second-countable T_0 space. The question this time is whether $\mathcal{O}(\omega^\omega)$ has universality in some sense. The following observation is a partial answer.

Proposition 2.12 (cf. [10, Proposition 7.14])**.** *If X is a quasi-Polish space, then $\mathcal{O}(X)$ is embedded into $\mathcal{O}(\omega^\omega)$.*

Although Proposition 2.12 has already been proved in [10, Proposition 7.14], we provide the complete proof to discuss its effective version. The key observation is that for any nonempty quasi-Polish space X there is an open continuous surjection from ω^ω onto X; see de Brecht [8, Lemma 38].

A computable function $f\colon X \to Y$ is *computably open* if $U \mapsto f[U]\colon \mathcal{O}(X) \to \mathcal{O}(Y)$ is computable.

Lemma 2.13. *Let X and Y be represented cb$_0$ spaces. Assume that there is a computable open surjection from X onto Y. Then $\mathcal{O}(Y)$ is computably embedded into $\mathcal{O}(X)$. In particular, the $\mathcal{O}(Y)$-degrees are included in the $\mathcal{O}(X)$-degrees.*

Proof. Fix a computable open surjection $\delta\colon X \to Y$. We show that $U \mapsto \delta^{-1}[U]$ gives a computable embedding of $\mathcal{O}(Y)$ into $\mathcal{O}(X)$. First note that computability of δ always implies computability of $\delta^{-1}\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ since the characteristic function of $\delta^{-1}[U]$ is just the composition $\chi_U \circ \delta\colon Y \to \mathbb{S}$; see e.g. [24]. It remains to show computability of $\delta^{-1}[U] \mapsto U$. Since δ is surjective, we have $\delta[\delta^{-1}[U]] = U$. By our assumption, δ is computably open, so $\delta^{-1}[U] \mapsto \delta[\delta^{-1}[U]] = U$ is computable. □

Proof of Proposition 2.12. As shown in de Brecht [8, Lemma 38], for any nonempty quasi-Polish space X there is an open continuous surjection from ω^ω onto X. Then, apply (the relativization of) Lemma 2.13. □

A represented cb$_0$ space is *computably quasi-Polish* if there is a computable open surjection $\delta\colon \omega \to X$; see [9]. Such a space is also called effectively quasi-Polish in [16]. Lemma 2.13 shows that if X is computably quasi-Polish then $\mathcal{O}(X)$ is computably embedded in to $\mathcal{O}(\omega^\omega)$. As a consequence, we get the following:

Corollary 2.14. *If X is a computable Polish space, then $\mathcal{O}(X)$ is computably embedded into $\mathcal{O}(\omega^\omega)$.*

In particular, the $\mathcal{O}(X)$-degrees are included in the $\mathcal{O}(\omega^\omega)$-degrees.

Proof. Let $X = (X, d, \alpha)$ be a computable Polish space; that is, $\{\alpha_i\}_{i\in\omega}$ is a dense subset of X, and $(i, j) \mapsto d(\alpha_i, \alpha_j)$ is computable. We define a system $(B_\sigma)_{\sigma\in\omega^{<\omega}}$ of rational open sets. Let $B_{\langle\rangle} = X$. Assume that B_σ has been already defined. Then, let $B_{\sigma^\frown n}$ be the n-th rational open ball B of radius less than 2^{-n} whose formal closure is contained in B_σ; that is, if $B_\sigma = B(\alpha_i; q)$ and $B = B(\alpha_j; r)$ then $d(\alpha_i, \alpha_j) < q - r$. This system (also known as the Suslin scheme; see [20, Exercise I.7.14]) generates a computable open surjection from ω^ω onto X. Then apply Lemma 2.13. □

Let us look at some examples of Polish spaces X for which the degree structure of $\mathcal{O}(X)$ is maximal.

Corollary 2.15. *The hyperspaces $\mathcal{O}(\mathbb{R}^\omega)$ and $\mathcal{O}(\omega^\omega)$ are computably bi-embeddable.*

In particular, the $\mathcal{O}(\mathbb{R}^\omega)$-degrees are exactly the $\mathcal{O}(\omega^\omega)$-degrees.

Proof. The hyperspace $\mathcal{O}(\mathbb{R}^\omega)$ is computably embedded into $\mathcal{O}(\omega^\omega)$ by Proposition 2.14. For the other direction, given σ, put $E_\sigma = \prod_{n<|\sigma|}(\sigma(n) - 2/3, \sigma(n) + 2/3)$. It is not hard to check that this yields a computable embedding of $\mathcal{O}(\omega^\omega)$ into $\mathcal{O}((-2/3, \infty)^\omega)$, which is computably homeomorphic to $\mathcal{O}(\mathbb{R}^\omega)$. □

In the case of $X = \mathcal{C}(\mathbb{R})$, the degree structure of $\mathcal{O}(X)$ is also maximal.

Proposition 2.16. *The hyperspace $\mathcal{O}(\mathbb{R}^\omega)$ and $\mathcal{O}(\mathcal{C}(\mathbb{R}))$ are computably bi-embeddable. In particular, the $\mathcal{O}(\mathcal{C}(\mathbb{R}))$-degrees are exactly the $\mathcal{O}(\omega^\omega)$-degrees.*

To show Proposition 2.16, we use the notion of a computable section-retraction pair. A space Y is called a *retract* of X if there are continuous functions $r\colon X \to Y$ and $s\colon Y \to X$ such that $r \circ s$ is the identity map on Y. Such an r is called a *retraction*, and s is called a *section*. If such r and s are computable, then Y is called a *computable retract* of X. The following is obvious (since a computable section is especially a computable embedding).

Observation 2.17. *If Y is a computable retract of X, then Y computably embeds into X, and in particular, every Y-degree is an X-degree.*

We also need the following fact:

Fact 1 (cf. Hoyrup [14])**.** *If X_0 and Y_0 are computable retracts of X_1 and Y_1, then $C(X_0, Y_0)$ is a computable retract of $C(X_1, Y_1)$.*

Proof of Proposition 2.16. We first see that $\mathbb{R}^\omega$ is a computable retract of $C(\mathbb{R})$. For $f \in \mathbb{R}^\omega$, define $s(f)\colon \mathbb{R} \to \mathbb{R}$ to be a piecewise linear map extending $f\colon \omega \to \mathbb{R}$, and for $g \in C(\mathbb{R})$, consider $r(g)\colon n \mapsto g(n)$. It is easy to check that (s, r) is a computable section-retraction pair. Hence, by Fact 1 (for $Y_0 = Y_1 = \mathbb{S}$), $\mathcal{O}(\mathbb{R}^\omega)$ is a computable retract of $\mathcal{O}(\mathcal{C}(\mathbb{R}))$. In particular, by Observation 2.17, the hyperspace $\mathcal{O}(\mathbb{R}^\omega)$ is computably embeddable into $\mathcal{O}(\mathcal{C}(\mathbb{R}))$. □

2.2.2. *The hyperspace on the function space with cofinite topology.* As we have already seen, if X is a computable quasi-Polish space, the degree structure of $\mathcal{O}(X)$ is contained in that of $\mathcal{O}(\omega^\omega)$. Here, we look at an example of the degree structure of the hyperspace on a function space which is not quasi-Polish.

Theorem 2.18. *The $\mathcal{O}\left(C(\omega_{\rm cof})\right)$-degrees are exactly the $\mathcal{O}(\omega^\omega)$-degrees.*

The proof is divided into two propositions below.

Proposition 2.19. *Each $\mathcal{O}\left(C(\omega_{\rm cof})\right)$-degree is a $\mathcal{O}(\omega^\omega)$-degree.*

Proof. Recall that for each $U \in \mathcal{O}(\omega^\omega)$, a local network at U is any set N such that $U = \bigcup_{\sigma \in N}[\sigma]$. As usual we identify σ with its code. For each $W \in \mathcal{O}\left(C(\omega_{\rm cof})\right)$, a local network at W is any set M containing codes of finite sets such that $W = \bigcup_{F \in M} \bigcap_{\langle n,D\rangle \in F} \left\{g \in \omega^\omega \mid g^{-1}\{n\} \subseteq D\right\}$. We write

$$[n \lhd D] = \{g \in \omega^\omega \mid g^{-1}\{n\} \subseteq D\};$$

see also Definition 3.16 for a related definition. F is said to be *good* if for each n there is at most one D such that $\langle n, D\rangle \in F$, and if $\langle n, D\rangle \in F$ and $\langle n', D'\rangle \in F$ and $n \neq n'$ then $D \cap D' = \emptyset$. A local network N is good if every $F \in N$ is good. It is easy to see that there is a single enumeration operator Δ such that if N is a local network at some $W \in \mathcal{O}\left(C(\omega_{\rm cof})\right)$, then Δ^N is a good local network at the same open set W. For this reason we will concern ourselves in this proof with only the good local networks of $\mathcal{O}\left(C(\omega_{\rm cof})\right)$.

In order to show that each $\mathcal{O}\left(C(\omega_{\rm cof})\right)$-degree is a $\mathcal{O}(\omega^\omega)$-degree, we shall need to interpret basic open sets of $\mathcal{O}\left(C(\omega_{\rm cof})\right)$ inside $\mathcal{O}(\omega^\omega)$. We define a labeling ℓ of all finite strings of positive length by the following. Suppose ℓ has been defined on every proper initial segment of σ of positive length. Now enumerate all finite sets disjoint from $\bigcup_{0<k<|\sigma|} \ell\left(\sigma \upharpoonright k\right)$. Let $\ell(\sigma)$ be the $\sigma\left(|\sigma|-1\right)^{th}$ finite set in this enumeration. (As usual we identify each finite set with its canonical index). Given $\sigma \in \omega^{<\omega}$ of positive length, we identify σ with the finite set $F_\sigma = \{\langle 0, \ell(\sigma \upharpoonright 1)\rangle, \langle 1, \ell(\sigma \upharpoonright 2)\rangle, \cdots, \langle |\sigma|-1, \ell(\sigma)\rangle\}$. Note that each F_σ is good.

Define the enumeration operators Φ and Ψ by the following. If N is a local network at some $U \in \mathcal{O}(\omega^\omega)$, we let $\Phi^N = \{F_\sigma \mid \sigma \in N\}$. If M is a local network at some $W \in \mathcal{O}\left(C(\omega_{\rm cof})\right)$, we let $\Psi^M = \bigcup_{F \in M} \bigcup\{\sigma \in \omega^{<\omega} \mid \bigcap_{\langle n,D\rangle \in F_\sigma}[n \lhd D] \subseteq \bigcap_{\langle n,D\rangle \in F}[n \lhd D]\}$. Notice that the "$\subseteq$" condition in the line above can be checked computably given F and σ.

To complete the proof of Proposition 2.19, we shall prove the following three statements:

(i) If N and $\hat{N}$ are local networks at the same $U \in \mathcal{O}(\omega^\omega)$, and U has the "downwards closed" property that $[\sigma] \subseteq U$ and $\bigcap_{\langle n,D\rangle \in F_\tau}[n \lhd D] \subseteq$

$\bigcap_{\langle n,D\rangle\in F_\sigma}[n\lhd D]$ implies that $[\tau]\subseteq U$, then Φ^N and $\Phi^{\hat{N}}$ are local networks at the same $W\in\mathcal{O}\left(C(\omega_{\mathrm{cof}})\right)$.

(ii) If M and $\hat{M}$ are local networks at the same $W\in\mathcal{O}\left(C(\omega_{\mathrm{cof}})\right)$, then Ψ^M and $\Psi^{\hat{M}}$ are local networks at the same $U\in\mathcal{O}(\omega^\omega)$.

(iii) For every good local network M at $W\in\mathcal{O}\left(C(\omega_{\mathrm{cof}})\right)$, $\Phi(\Psi(M))$ is a local network at W.

First we check that conditions (i) to (iii) imply that each $\mathcal{O}\left(C(\omega_{\mathrm{cof}})\right)$-degree is a $\mathcal{O}(\omega^\omega)$-degree. Let $W\in\mathcal{O}\left(C(\omega_{\mathrm{cof}})\right)$. Fix any good local network M at W. We claim that $\mathtt{Name}(W)\equiv_M \mathtt{Name}(U)$, where U is generated by Ψ^M. Clearly, Ψ will reduce every local network of W to a local network of U, thus $\mathtt{Name}(W)\geq_M \mathtt{Name}(U)$. Now let $\hat{N}=\Psi^M$ and consider a local network N of U. It is easy to see that U has the "downwards closed" property assumed in (i), by examining the definition of Ψ^M. Applying (i) and (iii) we get that Φ^N is a local network at W. This means that $\mathtt{Name}(W)\leq_M \mathtt{Name}(U)$, and so W has the same degree as $U\in\mathcal{O}(\omega^\omega)$.

We now prove statement (i). Let N and $\hat{N}$ be local networks at the same $U\in\mathcal{O}(\omega^\omega)$, where U has the "downwards closed" property. It suffices to show that

$$\bigcup_{F\in\Phi^N}\bigcap_{\langle n,D\rangle\in F}[n\lhd D]\subseteq\bigcup_{\hat{F}\in\Phi^{\hat{N}}}\bigcap_{\langle\hat{n},\hat{D}\rangle\in\hat{F}}[\hat{n}\lhd\hat{D}].$$

We fix some $F\in\Phi^N$ and some function $g\in\bigcap_{\langle n,D\rangle\in F}\mathcal{G}^+(D,n)$. Let $\sigma\in N$ be such that $F=F_\sigma$. Let $\alpha\in\omega^\omega$ be defined to be such that $\ell(\alpha(n))=g^{-1}\{n\}$ for all n; since g is a function, $g^{-1}\{n\}$ is disjoint from $\bigcup_{k<n}g^{-1}\{k\}$. Since U has the "downwards closed" property, by considering $\alpha\restriction|\sigma|$, we see that $\alpha\supset\hat{\sigma}$ for some $\hat{\sigma}\in\hat{N}$. This means that $g\in\bigcap_{\langle\hat{n},\hat{D}\rangle\in F_{\hat{\sigma}}}[\hat{n}\lhd\hat{D}]$.

Now we prove statement (ii). Suppose M and $\hat{M}$ are local networks at the same $W\in\mathcal{O}\left(C(\omega_{\mathrm{cof}})\right)$. Suppose σ is chosen so that $\bigcap_{\langle n,D\rangle\in F_\sigma}[n\lhd D]\subseteq\bigcap_{\langle n,D\rangle\in F}[n\lhd D]$ for some $F\in M$. We shall need to see that $[\sigma]\subset\left[\Psi^{\hat{M}}\right]$. Let $\alpha\in[\sigma]$. Define g so that $g^{-1}\{n\}=\ell\left(\alpha\restriction n\right)\cup\{z_{n-|\sigma|}\}$, where z_m is the m^{th} element not in $\bigcup_{k>0}\ell\left(\alpha\restriction k\right)$. Obviously if $z_{n-|\sigma|}$ is not defined then we take an empty union. Clearly g is total and well-defined as a function. Note that $g\in\bigcap_{\langle n,D\rangle\in F_\sigma}[n\lhd D]\subseteq\bigcap_{\langle n,D\rangle\in F}[n\lhd D]$. Since $F\in M$, this means that $\bigcap_{\langle n,D\rangle\in F}[n\lhd D]\subseteq\bigcup_{\hat{F}\in\hat{M}}\bigcap_{\langle\hat{n},\hat{D}\rangle\in\hat{F}}[\hat{n}\lhd\hat{D}]$, and so we can fix some $\hat{F}\in\hat{M}$ such that $g\in\bigcap_{\langle\hat{n},\hat{D}\rangle\in\hat{F}}[\hat{n}\lhd\hat{D}]$. By the definition of g, this certainly means that $\bigcap_{\langle n,D\rangle\in F_{\alpha\restriction m}}[n\lhd D]\subseteq\bigcap_{\langle\hat{n},\hat{D}\rangle\in\hat{F}}[\hat{n}\lhd\hat{D}]$ for a large enough m. Thus, $\alpha\in\left[\Psi^{\hat{M}}\right]$.

Finally we show (iii). Let $F\in M$. Let S be the set $\{\sigma\in\omega^{<\omega}\mid\bigcap_{\langle n,D\rangle\in F_\sigma}[n\lhd D]\subseteq\bigcap_{\langle n,D\rangle\in F}[n\lhd D]\}$. Obviously, $\Phi(S)$ generates an open set which is contained in the basic open set generated by F. Conversely, since F is good, by considering all strings in S of a certain fixed length, we certainly have that the basic open set generated by F is covered by the open set generated by F_σ for all $\sigma\in S$ of this fixed length. $\square$

Proposition 2.20. *Each $\mathcal{O}(\omega^\omega)$-degree is a $\mathcal{O}\left(C(\omega_{\mathrm{cof}})\right)$-degree.*

Proof. Given $U\in\mathcal{O}(\omega^\omega)$ we construct $W\in\mathcal{O}(C(\omega_{\mathrm{cof}}))$. Let N be a local network at U; that is, $U=[N]$, where $[N]=\bigcup_{\sigma\in N}[\sigma]$. We construct an enumeration operator Φ such that $\Phi(N)$ yields a local network at W. To each $\sigma \in \omega^{<\omega}$ assign a finite set $D_\sigma \subseteq \omega$:

$$D_{\langle\rangle} = \{0\}, \qquad\qquad D_{\sigma^\frown k} = D_\sigma \cup \{|\sigma|, r_k\},$$

where r_k is the kth element in $\omega \setminus (D_\sigma \cup \{|\sigma|\})$. For instance, $D_{\langle a\rangle} = \{0, 1, a+2\}$. Then put $p(\sigma) = \bigcap_{n\leq|\sigma|}[n \lhd D^*_{\sigma\restriction n}]$, where $D^*_\sigma = D_\sigma \setminus D_{\sigma\restriction|\sigma|-1}$. So, $p(\sigma)$ is the sequence of declarations $\mathbf{f}^{-1}\{n\} \subseteq D^*_{\sigma\restriction n}$ for each $n \leq |\sigma|$.

A finite tree $H \subseteq \omega^{<\omega}$ is *homogeneous* if the set of all leaves of H is of the form $\prod_{n<\ell} H_n$ for some finite $H_n \subseteq \omega$. Put $H \restriction n = \{\tau \in \omega^{\leq n} : (\forall k < n)\ \tau(k) \in H_k\}$; then $H = H \restriction \ell$. Then $\Phi(N)$ enumerates basic open sets in $C(\omega_{\rm cof})$ in the following manner: For any homogeneous tree H generated by $(H_k)_{k<\ell}$, consider the following basic open set in $C(\omega_{\rm cof})$:

$$\tilde{p}(H) = \bigcap_{k<\ell}\bigcap_{\sigma\in H\restriction k}\left[k \lhd \bigcup\{D_{\sigma^\frown i} : i \in H_k\}\right].$$

If we see $[H] \subseteq [N_s]$ for finite $N_s \subseteq N$, enumerate $\tilde{p}(H)$ into $\Phi(N)$. Note that $\downarrow\!\sigma := \{\tau : \tau \preceq \sigma\}$ is a homogeneous tree, and $\tilde{p}(\downarrow\!\sigma) = \bigcap_{n<|\sigma|}[n \lhd D_{\sigma\restriction n}]$. Hence, $p(\sigma) \subseteq \tilde{p}(\downarrow\!\sigma)$. If $\sigma \in N$ then $[\downarrow\!\sigma] = [\{\sigma\}]$ and $\{\sigma\} \subseteq N$, so $p(\sigma)$ is enumerated into $\Phi(N)$. Moreover, enumerate $[0 \lhd \{1\}]$ and $[1 \lhd \{0\}]$ into $\Phi(N)$. Note that what is enumerated by $\Phi(N)$ is a local network at the union $W \in \mathcal{O}(C(\omega_{\rm cof}))$ of all such basic open sets. Then define $\hat{\Phi}(N)$ to be such W.

As $[0 \lhd \{1\}], [1 \lhd \{0\}] \subseteq \hat{\Phi}(N)$, if either $f^{-1}\{0\} \subseteq \{1\}$ or $f^{-1}\{1\} \subseteq \{0\}$ then $f \in \hat{\Phi}(N)$. Note that, if f is a constant function, we have $f^{-1}\{n\} = \emptyset$ for all but one n; in particular, either $f^{-1}\{0\} = \emptyset$ or $f^{-1}\{1\} = \emptyset$. This shows that $\hat{\Phi}(N)$ contains all constant functions.

Claim. If $[M] \subseteq [N]$ then $\hat{\Phi}(M) \subseteq \hat{\Phi}(N)$.

Proof. Let $f \in \hat{\Phi}(M)$. Then, by the definition of Φ, there is a homogeneous tree $[H] \subseteq [M]$, and $f \in \tilde{p}(H)$. If f is constant, we have $f \in \hat{\Phi}(N)$ as mentioned above. Thus, we may assume that f is finite-to-one.

As f is finite-to-one, one can construct a finitely branching infinite homogeneous tree H^∞ such that $[H^\infty] \subseteq [H]$ and $f \in \tilde{p}(H^\infty \restriction k)$ for any k, where $[H^\infty]$ is the set of all infinite paths through H^∞.

To see this, assume that H is generated by $(H_k)_{k<\ell}$. First note that $f \in \tilde{p}(H)$ implies $f \in \tilde{p}(H \restriction m)$ for any $m \leq \ell$. So, we consider $m \geq \ell$. Inductively assume that $(H_k)_{k<m}$ is constructed, and $f \in \tilde{p}(H \restriction m)$. Since $f^{-1}\{m\}$ is finite, by choosing a sufficiently large finite set $H_m \subseteq \omega$, we may ensure $f^{-1}\{m\} \subseteq \bigcup\{D_{\sigma^\frown i} : i \in H_m\}$ for any $\sigma \in H \restriction m$. This implies $f \in \tilde{p}(H \restriction m+1)$. Continuing this procedure, we get a sequence $(H_k)_{k\in\omega}$ of finite subsets of ω, which generates a tree H^∞ such that $[H^\infty] = \prod_k [H_k]$.

Now, we have $[H^\infty] \subseteq [H] \subseteq [M] \subseteq [N]$, so we find ℓ such that $[H^\infty \restriction \ell] \subseteq [N_s]$ at some finite $N_s \subseteq N$ by König's lemma. Thus, $f \in \tilde{p}(H^\infty \restriction \ell) \subseteq \hat{\Phi}(N)$. $\square$

The above claim shows that our construction $U \mapsto W$ does not depend on the choice of a local network N at U. Thus, we write $\varphi(U)$ for $\hat{\Phi}(N)$, which is well-defined.

Conversely, given $W \in \mathcal{O}(C(\omega_{\rm cof}))$, let V be a local network at W. Then $\Psi(V)$ enumerates all σ such that $p(\sigma)$ is included in some basic open set enumerated into V. We show that if $W = \varphi(U)$, then $[\Psi(V)] = U$.

Given any $\alpha \in \omega^\omega$, define $f_\alpha \in C(\omega_{\rm cof})$ by $f_\alpha^{-1}\{n\} = D^*_{\alpha\upharpoonright n}$. In particular, $f_\alpha^{-1}\{0\} = \{0\}$ and $f_\alpha^{-1}\{1\} = \{1, \alpha(0)+2\}$. Obviously, we have $|f_\alpha^{-1}\{n\}| \leq 2$ for each n, and f_α is total since we require that $n \in D_{\alpha\upharpoonright n}$ for any n. Note also that $\alpha \neq \beta$ implies $f_\alpha \neq f_\beta$ since $r_j \in D_{\sigma^\frown j} \neq D_{\sigma^\frown k} \ni r_k$ whenever $j \neq k$.

Claim. If $W \subseteq \varphi(U)$, then $[\Psi(V)] \subseteq U$.

Proof. Assume $\alpha \in [\Psi(V)]$. Then $\alpha \in [\sigma]$ for some σ which is enumerated into $\Psi(V)$. By our construction, this happens only if $p(\sigma)$ is included in some basic open set in V. Note that $f_\alpha \in p(\sigma) \subseteq W$ since $f_\alpha^{-1}\{n\} = D^*_{\sigma\upharpoonright n}$ for any $n < |\sigma|$. Let N be a local network at U. By our assumption, we have $f_\alpha \in W \subseteq \varphi(U) = \hat{\Phi}(N)$. As $f_\alpha^{-1}\{0\} = \{0\}$ and $f_\alpha^{-1}\{1\} = \{1, \alpha(0)+2\}$, we have $f^{-1}\{0\} \not\subseteq \{1\}$ and $f^{-1}\{1\} \not\subseteq \{0\}$; that is, $f \notin [0 \lhd \{1\}]$ and $f \notin [1 \lhd \{0\}]$. Thus, the condition $f_\alpha \in \hat{\Phi}(N)$ must be witnessed by $f_\alpha \in \tilde{p}(H)$ for some homogeneous tree $[H] \subseteq [N]$. This means that $f_\alpha^{-1}\{k\} \subseteq \bigcup\{D_{\sigma^\frown i} : i \in H_k\}$ for any $\sigma \in H \upharpoonright k$, where assume that H is generated by $(H_k)_{k<\ell}$.

By induction, we show that $\alpha \upharpoonright k \in H \upharpoonright k$ for any $k \leq \ell$. If $\alpha \upharpoonright k \in H \upharpoonright k$ then we must have $f_\alpha^{-1}\{k\} \subseteq \bigcup\{D_{(\alpha\upharpoonright k)^\frown i} : i \in H_k\}$. Recall that $f_\alpha^{-1}\{k\}$ is of the form $\{k, r_{\alpha(k)}\}$, and moreover, $r_{\alpha(k)} \in D_{(\alpha\upharpoonright k)^\frown i}$ then $i = \alpha(k)$. Thus, we get $\alpha(k) \in H_k$. Hence, $\alpha \upharpoonright \ell \in H$, and so $\alpha \in [H] \subseteq [N] = U$. $\square$

Claim. If $\varphi(U) \subseteq W$, then $U \subseteq [\Psi(V)]$.

Proof. Let $\alpha \in U$. Fix any local network N at U, i.e., $U = \bigcup_{\sigma\in N}[\sigma]$. Then $\alpha \in [\sigma]$ for some $\sigma \in N$. As mentioned above, $p(\sigma) \subseteq \tilde{p}(\downarrow\sigma)$ is enumerated into $\Phi(N)$. We also have $f_\alpha \in p(\sigma)$ since α extends σ. Thus, $f_\alpha \in \hat{\Phi}(N) = \varphi(U) \subseteq W$ by our assumption. If V is a local network at W then V must enumerate some basic open set containing f_α, say $\bigcap_{n\in I}[n \lhd E_n]$ for a finite collection $(E_n)_{n\in I}$ of finite sets. This means $f_\alpha^{-1}\{n\} = D^*_{\alpha\upharpoonright n} \subseteq E_n$; hence $[n \lhd D^*_{\alpha\upharpoonright n}] \subseteq [n \lhd E_n]$, for each $n \in I$. Then, for $m = \max I$, we get $p(\alpha \upharpoonright m) \subseteq \bigcap_{n\in I}[n \lhd E_n]$, so $\alpha \upharpoonright m$ is enumerated into $\Psi(V)$ by our construction. Consequently, $\alpha \in [\Psi(V)]$. $\square$

Now, define $\psi(W) = [\Psi(V)]$ (which may be ill-defined, in general). If W is of the form $\varphi(U)$, by the above claims, $U = [\Psi(V)] = \psi(W)$ independent of the choice of V. So, in this case, $\psi(W)$ is well-defined. Consequently, we get $\psi(\varphi(U)) = U$ for any $U \in \mathcal{O}(\omega^\omega)$. As φ and ψ are computable, this concludes that $U \in \mathcal{O}(\omega^\omega)$ is $\equiv_T$-equivalent to $\varphi(U) \in \mathcal{O}(C(\omega_{\rm cof}))$. $\square$

2.2.3. *The hyperspace on the holistic space.* Here, we give a few side results. The holistic space is a represented topological space introduced as a technical tool for characterizing continuous degrees [2]. A holistic set is any set $H \subseteq \omega^\omega$ satisfying the following properties. For every $\sigma \in \omega^{<\omega}$,

- $\sigma^\frown(2n)$ and $\sigma^\frown(2n+1)$ are not both in H, for every n.
- If $\sigma \notin H$ then $\sigma^\frown(2n) \in H$ for every n.
- If $\sigma \in H$ then $\sigma^\frown(2n+1) \in H$ for some n.

The holistic space $\mathcal{H}$ is the space of all holistic sets, with the subbasis consisting of $B^{\mathcal{H}}_\sigma = \{H \in \mathcal{H} \mid \sigma \in H\}$, where σ is a finite string. If $\Gamma = \{\sigma_0, \cdots, \sigma_k\}$ is a finite set of finite strings, we let $B^{\mathcal{H}}_\Gamma$ denote $\bigcap_{i\leq k} B^{\mathcal{H}}_{\sigma_i}$. Notice that given Γ_0 and Γ_1, it is computable to check if $B^{\mathcal{H}}_{\Gamma_0} \subseteq B^{\mathcal{H}}_{\Gamma_1}$. A local network of an element $W \in \mathcal{O}(\mathcal{H})$ is a set M of finite sets of finite strings such that $W = \bigcup_{\Gamma\in M} B^{\mathcal{H}}_\Gamma$.

Theorem 2.21. *The $\mathcal{O}(\mathcal{H})$-degrees are exactly the $\mathcal{O}(\omega^\omega)$-degrees.*

Proof. Notice that $\mathcal{H}$ is a computable Polish space, so by Proposition 2.14, every $\mathcal{O}(\mathcal{H})$-degree is a $\mathcal{O}(\omega^\omega)$-degree. We now prove the other direction.

Given any $\sigma \in \omega^{<\omega}$ we let $\varphi(\sigma)$ be defined to be the string τ such that $|\tau| = |\sigma|$, and for every $n < |\sigma|$ we have $\tau(n) = 2\sigma(n)+1$. We now define the enumeration operators Φ and Ψ as follows. Given a set $N \subseteq \omega^{<\omega}$ we let $\Phi(N) = \{\{\varphi(\sigma)\} \mid \sigma \in N\}$. The definition of Ψ is slightly more involved. First, we call a finite set $D \subseteq \omega^{<\omega}$ *extendible* if there is a holistic set $H \supset D$. It is not hard to see ([2, Lemma 5.3]) that the class of all extendible sets is computable. Since $B^{\mathcal{H}}_D \neq \emptyset$ if and only if D is extendible, we will restrict ourselves to those basic open sets generated by an extendible D.

Definition of $H(\delta, D)$. We shall require another definition. Suppose that δ is an infinite string, and D is a finite set of finite strings such that $D \cap L = \emptyset$ and $D \cup \{\varphi(\delta \upharpoonright i) \mid i \in \omega\}$ is consistent for every i. We shall define the set $H(\delta, D) \subseteq \omega^{<\omega}$ by the following. First we put $\langle\rangle \in H(\delta, D)$. Now if $\alpha \notin H(\delta, D)$, we let $\alpha^\frown n \in H(\delta, D)$ if and only if n is even. Suppose that $\alpha \in H(\delta, D)$. For each n such that there is a sequence (possibly empty) of odd numbers $o_0, \cdots, o_k$ such that $\alpha^\frown n^\frown o_0{}^\frown \cdots {}^\frown o_k \in D$, we put $\alpha^\frown n \in H(\delta, D)$. Furthermore, if $\alpha = \varphi(\delta \upharpoonright i)$ then we also put $\varphi(\delta \upharpoonright i)^\frown(2\delta(i) + 1)$ into $H(\delta, D)$. Otherwise if $\alpha \notin L$ we put $\alpha^\frown(2m + 1) \in H(\delta, D)$ for some m larger than all n (if n does not exist, we take $m = 0$). Include no other successors of α. We now claim that $H(\delta, D)$ is holistic, $H(\delta, D) \supseteq D$ and $H(\delta, D) \cap L = \{\varphi(\delta \upharpoonright i) \mid i \in \omega\}$.

To see that $H(\delta, D)$ is holistic we need to check that there are no α and k such that $\alpha^\frown(2k) \in H(\delta, D)$ and $\alpha^\frown(2k + 1) \in H(\delta, D)$. The only way this can happen is if $\alpha \in H(\delta, D)$. If $\alpha \notin L$ then it must be that $\alpha^\frown(2k)^\frown o_0{}^\frown \cdots {}^\frown o_k \in D$ and $\alpha^\frown(2k+1)^\frown o'_0{}^\frown \cdots {}^\frown o'_{k'} \in D$ for some sequences of odd numbers. However it is clear that if H is a holistic set and $\beta^\frown(2i+1) \in H$ then $\beta \in H$. Hence if H is any holistic set such that $H \supseteq D$, both $\alpha^\frown(2k)$ and $\alpha^\frown(2k+1)$ must be in H, and D cannot be consistent. On the other hand, if $\alpha \in L$ then it must be that $\alpha^\frown(2k)^\frown o_0{}^\frown \cdots {}^\frown o_k \in D$ and $\alpha^\frown(2k+1) \subset \varphi(\delta)$, which means that $D \cup \{\varphi(\delta \upharpoonright (|\alpha| + 1)\}$ is not consistent. In any case, $H(\delta, D)$ is holistic.

Now we check that $H \supseteq D$. Let $\alpha^\frown(2n)^\frown o_0{}^\frown \cdots {}^\frown o_k \in D$ for some n and odd integers $o_0, \cdots o_k$. (Note that every element of D is of this form since $D \cap L = \emptyset$). The construction ensures that $\alpha^\frown(2n) \in H(\delta, D)$ (this is true even if $\alpha \notin H(\delta, D)$). But this means that $\alpha^\frown(2n)^\frown o_0{}^\frown \cdots {}^\frown o_i \in H(\delta, D)$ for every $i \leq k$. Finally we verify that $H(\delta, D) \cap L = \{\varphi(\delta \upharpoonright i) \mid i \in \omega\}$. Suppose $\alpha^\frown(2n + 1) \in H(\delta, D) \cap L$ (the other direction is obvious). Then as $H(\delta, D)$ is holistic, this means that $\alpha \in H(\delta, D) \cap L$, and so we may assume that $\alpha = \varphi(\delta \upharpoonright i)$ for some i. If $n \neq \delta(i)$ then this means that $\alpha^\frown(2n + 1)^\frown o_0{}^\frown \cdots {}^\frown o_k \in D$ for some odd sequence. Hence $D \cap L \neq \emptyset$, a contradiction.

Defining Ψ. Given a consistent $D \subseteq \omega^{<\omega}$, we define $\psi(D)$ by the following. If D contains two incomparable elements of L we set $\psi(D) = \emptyset$ (here $L = \varphi[\omega^{<\omega}]$). Otherwise let α be the longest element of $L \cap D$, if it exists, or let $\alpha = \langle\rangle$ if $D \cap L = \emptyset$. Let $\psi(D)$ contain all σ such that $\varphi(\sigma) \supseteq \alpha$ and $|\varphi(\sigma)| > \max\{|\tau| \mid \tau \in D\}$, and such that $D \cup \{\varphi(\sigma)\}$ is consistent. Finally given $M \subseteq [\omega^{<\omega}]^{<\omega}$ we let $\Psi(M) = \bigcup\{\psi(D) \mid D \in M\}$.

Verification. Now we fix an open set $U \in \mathcal{O}(\omega^\omega)$, and with some local network N. Let $W \in \mathcal{O}(\mathcal{H})$ be the open set generated by $\Phi(N)$. We wish to argue that $\mathtt{Name}(W) \equiv_M \mathtt{Name}(U)$.

First of all, let $[\hat{N}] = [N] = U$, and fix some $\hat{\sigma} \in \hat{N}$ and some $H \in B^{\mathcal{H}}_{\varphi(\hat{\sigma})}$. We want to show that there exists some $\sigma \in N$ such that $H \in B^{\mathcal{H}}_{\varphi(\sigma)}$. Since $\varphi(\hat{\sigma}) \in H$, we can find some infinite string $\hat{\delta}$ such that $\left(\varphi(\hat{\sigma})^\frown\hat{\delta}\right) \upharpoonright i \in L \cap H$ for every i. (Notice that all proper prefixes of $\varphi(\hat{\sigma})$ must also be in $L \cap H$). But this means that there is some δ such that $\varphi\left((\hat{\sigma}^\frown\delta) \upharpoonright i\right) = \left(\varphi(\hat{\sigma})^\frown\hat{\delta}\right) \upharpoonright i$ for all i. Let i be such that $\sigma = (\hat{\sigma}^\frown\delta) \upharpoonright i \in N$. But this means that $H \in B^{\mathcal{H}}_{\varphi(\sigma)}$. We can apply the argument symmetrically to conclude that Φ witnesses that $\mathtt{Name}(W) \leq_M \mathtt{Name}(U)$.

Next, we fix a local network M of W. We want to verify that $[\Psi(M)] = [N] = U$, and hence $\mathtt{Name}(W) \geq_M \mathtt{Name}(U)$ via Ψ. We first argue that $[\Psi(M)] \subseteq [N]$. Let $\sigma \in \psi(D)$ for some consistent $D \in M$. Also let $\delta \supset \sigma$ be an infinite string. We wish to find some $\hat{\sigma} \in N$ such that $\hat{\sigma} \subset \delta$. Let $D' = D \setminus \{\alpha \in L \mid \alpha$ is comparable with $\varphi(\sigma)\}$. Hence $D' \cap L = \emptyset$. Since $D \cup \{\varphi(\sigma)\}$ is consistent, and $\varphi(\sigma)$ is longer than any string in D, we must have $D' \cup \{\varphi(\delta \upharpoonright i)\}$ is consistent for every i. Hence $H(\delta, D')$ can be constructed and has the property that it is holistic, $H(\delta, D') \supseteq D' \cup \{\varphi(\sigma)\}$ and $H(\delta, D') \cap L = \{\varphi(\delta \upharpoonright i) \mid i \in \omega\}$. Hence $H(\delta, D') \in B^{\mathcal{H}}_D \subseteq W = [\Phi(N)]$. Let $\hat{\sigma} \in N$ be such that $H(\delta, D') \in B^{\mathcal{H}}_{\{\varphi(\hat{\sigma})\}}$. As $\varphi(\hat{\sigma}) \in L$, this means that $\hat{\sigma} \subset \delta$.

Now we wish to see that $[N] \subseteq [\Psi(M)]$. Let $\sigma \subset \delta$ be such that $\sigma \in N$. Now $H(\delta, \emptyset) \in B^{\mathcal{H}}_{\{\varphi(\sigma)\}} \subseteq W$. Since M is a local network at W, let $D \in M$ be such that $H(\delta, \emptyset) \in B^{\mathcal{H}}_D$. Clearly, $\alpha \in D \cap L$ implies that $\alpha \in \{\varphi(\delta \upharpoonright i) \mid i \in \omega\}$. Since $D \cup \{\varphi(\delta \upharpoonright i) \mid i \in \omega\}$ is consistent for all i, so $\delta \upharpoonright i \in \psi(D)$ for a large enough i. This means that $\delta \in [\Psi(M)]$. □

2.3. **An intermediate degree structure between $\mathcal{O}(\omega)$ and $\mathcal{O}(\omega^\omega)$.** We show that the hyperspace $\mathcal{O}(\mathbb{Q})$ gives an intermediate degree structure between the e-degrees and $\mathcal{O}(\omega^\omega)$-degrees. Here, we consider $\mathbb{Q}$ as a subspace of $\mathbb{R}$.

Theorem 2.22. *The degree structure of the hyperspace $\mathcal{O}(\mathbb{Q})$ is strictly larger than that of $\mathcal{O}(\omega)$, but strictly smaller than that of $\mathcal{O}(\omega^\omega)$.*

This theorem is divided into several steps.

Lemma 2.23. *$\mathcal{O}(\mathbb{Q})$ contains all e-degrees.*

Proof. By Observation 2.17 and Fact 1, in order to show that the $\mathcal{O}(\omega)$-degrees are included into the $\mathcal{O}(\mathbb{Q})$-degrees, it suffices to check that ω is a computable retract of $\mathbb{Q}$. For each $n \in \omega$, choose an irrational z_n such that $z_n < n < z_{n+1}$. Given $q \in \mathbb{Q}$, one can effectively find the least n such that $z_n < |q| < z_{n+1}$. Then, define $r(q)$ to be such an n. Clearly, $r\colon \mathbb{Q} \to \omega$ is a computable retraction, and the identity map $n \mapsto n\colon \omega \to \mathbb{Q}$ yields a computable section. □

By Corollary 1.4, the $\mathcal{C}(\omega^\omega)$-degrees are included in the $\mathcal{O}(\omega^\omega)$-degrees since $\omega \times \omega^\omega \simeq \omega^\omega$. Thus, to see that the $\mathcal{O}(\mathbb{Q})$-degrees do not exhaust the whole $\mathcal{O}(\omega^\omega)$-degrees, it suffices to show that some $\mathcal{C}(\omega^\omega)$-degree is not an $\mathcal{O}(\mathbb{Q})$-degree.

In order to prove this, we need some technical notion. We say that a topological space X is Γ-*representable* (cf. Schröder-Selivanov [41]) if X has an admissible representation δ such that $\mathrm{Eq}(\delta) = \{(p, q) : p, q \in \mathrm{dom}(\delta) \text{ and } \delta(p) = \delta(q)\}$ is in $\Gamma(\omega^\omega)$.

Observation 2.24. *The hyperspace $\mathcal{O}(\mathbb{Q})$ is Π^0_3-representable.*

Proof. Observe that a local network at $U \in \mathcal{O}(\mathbb{Q})$ is a collection $(I_i)_{i\in\omega}$ of rational intervals such that $U = \bigcup_{i\in\omega} I_i$. If B_e is the eth rational interval, then a coded local network at $U \in \mathcal{O}(\mathbb{Q})$ is a collection of e's such that the union of B_e's is U. Recall that a name of $U \in \mathcal{O}(\mathbb{Q})$ is an enumeration of a local network at U; hence, it is a sequence $p \in \omega^\omega$ such that $U = \bigcup_{n\in\omega} B_{p(n)}$. Recall also that such a representation $p \mapsto U$ is admissible (see Section 1.2.6).

Clearly, $\delta\colon \omega^\omega \to \mathcal{O}(\mathbb{Q})$ is total. Given $p \in \omega^\omega$ and $e \in \omega$, $B_e \subseteq U_p$ if and only if every rational $r \in B_e \cap \mathbb{Q}$ is contained in U_p, which is a Π^0_2 property. Moreover, $U_p = U_q$ if and only if for any $e \in \omega$, the property $B_e \subseteq U_p$ is equivalent to that $B_e \subseteq U_q$. This is a Π^0_3 property. Consequently, $\mathcal{O}(\mathbb{Q})$ is Π^0_3-representable. □

We use the following fact (where a more general version will be proven later).

Fact 2 (see Theorem 3.21 below)**.** *If X is $\utilde{\mathbf{\Sigma}}^1_1$-representable, then there exists a $C(\omega^\omega)$-degree with is not an X-degree.*

Corollary 2.25. *There is a $\mathcal{C}(\omega^\omega)$-degree which is not an $\mathcal{O}(\mathbb{Q})$-degree.*

Proof. By Observation 2.24 and Fact 2. □

Next, we need to show that the $\mathcal{O}(\mathbb{Q})$-degrees are included in the $\mathcal{O}(\omega^\omega)$-degrees. Michael-Stone [30] showed that every analytic subset A of a Polish space is a quotient of the Baire space ω^ω. De Brecht et al. [10, Corollary 7.5] used this fact to show that if A is an analytic subset of a Polish space, then $\mathcal{O}(A)$ sequentially embeds into $\mathcal{O}(\omega^\omega)$. By effectivizing the result by de Brecht et al. [10, Theorem 7.4 and Proposition 7.6] which says that $\mathcal{O}(\mathbb{Q})$ sequentially embeds into $\mathcal{O}(\omega^\omega)$, one can show the following.

Lemma 2.26. *For any represented space Y, the function space $C(\omega^\omega, Y)$ contains all $\mathcal{C}(\mathbb{Q}, Y)$-degrees. In particular, the hyperspace $\mathcal{O}(\omega^\omega)$ contains all $\mathcal{O}(\mathbb{Q})$-degrees.*

Proof. If $A = \mathbb{Q}$, the proof described in [30] gives a computable quotient map $\delta\colon \omega^\omega \to \mathbb{Q}$. Then, by de Brecht et al. [10, Corollary 7.5], one can see that $\mathcal{C}(\mathbb{Q}, Y)$ computably embeds into $\mathcal{C}(\omega^\omega, Y)$. □

Finally, we will show that $\mathcal{O}(\mathbb{Q})$ contain a point which does not have an e-degree.

Lemma 2.27. *There is an $C(\mathbb{Q}, \omega)$-degree which is not an e-degree. In particular, both $C(\mathbb{Q})$ and $\mathcal{O}(\mathbb{Q})$ contain a point which does not have an e-degree.*

To make the notation simple, instead of the rationals $\mathbb{Q}$, we use the dyadic rationals $\mathbb{Q}_2$, the set of the form $n^\frown\sigma^\frown 0^\omega$ for some $n \in \omega$ and $\sigma \in 2^{<\omega}$. Note that $\mathbb{Q}_2 \subseteq \omega^\omega$ is a countable metric space with no isolated points, and therefore, homeomorphic to $\mathbb{Q}$.

Let $[\sigma]$ be the set of all infinite strings extending $\sigma \in \omega \times 2^{<\omega}$. We first note that the computable clopen basis $([\sigma])_{\sigma\in\omega\times 2^{<\omega}}$ of $\mathbb{Q}_2$ has the following property.

Observation 2.28. *In $\mathbb{Q}_2$, each basic clopen set $[\sigma]$ is partitioned into an infinite set of basic clopen sets.*

Proof. To see that $\mathbb{Q}_2$ has this property, fix σ. Choose a dyadic irrational α extending σ. Then, consider the set of all minimal strings in $\{\tau \succeq \sigma : \alpha \notin [\tau]\}$, and this yields an infinite sequence of disjoint basic clopen sets whose union covers $[\sigma]$ as $\alpha \notin \mathbb{Q}_2$. □

A basic neighborhood of the space $C(\mathbb{Q}_2,\omega)$ is of the form $[\sigma,n]=\{g\in C(\mathbb{Q}_2,\omega):[\sigma]\subseteq g^{-1}\{n\}\}$. Therefore, a name of $g\in C(\mathbb{Q}_2,\omega)$ is a sequence of declarations $[\sigma]\subseteq g^{-1}\{n\}$ which eventually determines g.

We now give a proof of Lemma 2.27. Note that, in the latter proof, one can replace $\mathbb{Q}_2$ with any space which has a computable clopen basis such that for each basic clopen set C one can effectively find an infinite partition of C into basic clopen sets.

Proof of Lemma 2.27. We will define $h\in C(\mathbb{Q}_2,\omega)$, by declaring a sequence $T=(T_k)$ of clopen sets in $\mathbb{Q}_2$ such that $h^{-1}\{k\}=T_k$.

Let $\langle\Phi_s,\Psi_s\rangle_{s\in\mathbb{N}}$ be a list of all pairs of enumeration operators, which yield partial computable functions $\varphi_s\colon\subseteq C(\mathbb{Q}_2,\omega)\to\mathbb{S}^\omega$ and $\psi_s\colon\subseteq\mathbb{S}^\omega\to C(\mathbb{Q}_2,\omega)$. At stage s, we attempt to ensure that $\varphi_s\psi_s(h)\neq h$ whenever $\psi_s(h)\in\mathbb{S}^\omega$. Assume that we have constructed a finite set $\Lambda_s\subseteq\omega$, and a collection $(T_t)_{t\in\Lambda_s}$ of clopen sets, where we also assume that $(T_t)_{t\in\Lambda_s}$ covers $\{[\langle r\rangle]\}_{r<s}$. Let u be the least element not in Λ_s, and choose a finite string σ_s such that $[\sigma_s]$ is not covered by $(T_t)_{t\in\Lambda_s}$.

Case 1. Assume that for any $g\in C(\mathbb{Q}_2,\omega)$ following $(T_t)_{t\in\Lambda_s}$ (i.e., $g^{-1}\{t\}=T_t$) and satisfying $[\sigma_s]\subseteq g^{-1}\{u\}$ for some $u\notin\Lambda_s$, if there is $D\subseteq\psi_s(g)$ such that $\langle\langle\tau,u\rangle,D\rangle\in\Phi_s$ (which declares $[\tau]\subseteq f^{-1}\{u\}$ for $f=\varphi_s(D)$), then τ does not extend σ_s.

In this case, if we ensure $h^{-1}\{u\}=[\sigma_s]$ for a large $u\notin\Lambda_s$, then for $f=\varphi_s\psi_s(h)$, we have $f^{-1}\{u\}\neq[\sigma_s]$. This is because for any declaration $[\tau]\subseteq f^{-1}\{u\}$ by Φ_s, we always have $[\tau]\not\subseteq[\sigma_s]$, and $f^{-1}\{u\}$ must be the union of such $[\tau]$'s. Thus, $h\neq\varphi_s\psi_s(h)$.

Hence, choose a large $u\notin\Lambda_s$, and put $T_u=[\sigma_s]$. If $\langle s\rangle$ is not covered by $(T_t)_{t\in\Lambda_s}$, choose a large number $v\notin\Lambda_s$, and define $T_v=\langle s\rangle\setminus\bigcup_{t\in\Lambda_s}T_t$, which is a clopen set as Λ_s is finite. Put $\Lambda_{s+1}=\Lambda_s\cup\{u,v\}$.

Case 2. Otherwise, there are $g\in C(\mathbb{Q}_2,\omega)$ following $(T_t)_{t\in\Lambda_s}$ and $[\sigma_s]\subseteq g^{-1}\{u\}$ with $u\notin\Lambda_s$ and some $D\subseteq\psi_s(g)$ such that $\langle\langle\tau,u\rangle,D\rangle\in\Phi_s$ and τ extends σ_s.

In this case, choose such a g. By the property of a clopen basis of $\mathbb{Q}_2$ mentioned above, $[\tau]$ can be partitioned into an infinite set of basic clopen sets. Moreover, the open set $g^{-1}\{u\}\setminus[\tau]$ can also be written as a countable union of basic clopen sets. In particular, there is an infinite increasing sequence (C_n) of clopen sets such that

$$C_0\subsetneq C_1\subsetneq C_2\subsetneq\cdots\to g^{-1}\{u\},$$

and $[\tau]\not\subseteq C_n$ for any $n\in\omega$, where the arrow indicates that $\bigcup_n C_n=g^{-1}\{u\}$. Consider a name of g given by this slowly converging sequence (C_n). Then a finite initial segment of such a name of g already witnesses $D\subseteq\psi_s(g)$. So, except for information on Λ_s, finitely many information $(T_t)_{t\in F}$ is used to ensure $D\subseteq\psi_s(g)$, where $[\tau]\not\subseteq[T_u]$. By extending a partial name, we can assume that $(T_t)_{t\in\Lambda_s\cup F}$ is pairwise disjoint. We choose such a $(T_u)_{u\in F}$.

Hence, choose a large $v\notin\Lambda_s\cup F$, and define T_v and Λ_{s+1} in the same manner as Case 1. Note that if we ensure that h follows $(T_t)_{t\in\Lambda_{s+1}}$, then for $f=\varphi_s\psi_s(h)$, $[\tau]\subseteq f^{-1}\{u\}$, but $[\tau]\not\subseteq h^{-1}\{u\}$; hence $h\neq\varphi_s\psi_s(h)$. This verifies the first assertion that there exists a $C(\mathbb{Q},\omega)$-degree which is not an e-degree.

For the second assertion, note that $C(\mathbb{Q},\omega)$ is computably bi-embeddable to $C(\mathbb{Q})$ since $C(\mathbb{Q},\omega)\subseteq C(\mathbb{Q})$, and $C(\mathbb{Q})$ is computably embedded into $C(\mathbb{Q},\omega^\omega)\simeq C(\mathbb{Q}\times\omega,\omega)\simeq C(\mathbb{Q},\omega)$. Therefore, $C(\mathbb{Q})$ has the same degree structure as $C(\mathbb{Q},\omega)$, which can be computably embedded in to $\mathcal{O}(\mathbb{Q})$ by Corollary 1.4. This completes the proof. $\square$

2.4. **Beyond $\mathcal{O}(\omega^\omega)$.** It is known that every second-countable T_0 space embeds into $\mathcal{O}(\omega)$. It is natural to ask if, whenever X and Y are second-countable and T_0, $\mathcal{C}(X, Y)$ can be embedded into $\mathcal{O}(\omega^\omega)$. By Corollary 1.4, the function space $C(X, Y)$ embeds into $\mathcal{O}(\omega \times X)$. As $\omega \times X$ is also second-countable, to answer the question, we only need to consider if $\mathcal{O}(X)$ embeds into $\mathcal{O}(\omega^\omega)$.

The key notion is the hierarchy of higher-order function spaces. For example, ω is the space of type 0 functions, ω^ω is the space of type 1 functions, and $C(\omega^\omega, \omega)$ is the space of type 2 functions, which can be embedded in $\mathcal{O}(\omega^\omega)$. This construction of function spaces can be extended to higher types. To be explicit, the hierarchy of function spaces is constructed as follows.

$$\omega\langle 0\rangle = \omega, \qquad\qquad \omega\langle k+1\rangle = C(\omega\langle k\rangle, \omega).$$

For example, $\omega\langle 2\rangle = C(\omega^\omega, \omega)$. These high-order function spaces are known as the *Kleene-Kreisel spaces*, and their computability-theoretic properties have been deeply studied; see [33, 25]. Here, note that the function space of type 2 or higher is not second-countable.

There is a method for obtaining a second-countable space from a complicated space. No matter what a represented space we consider, the space of all names of points is a subspace of ω^ω, so it is particularly second-countable. It is expected that the space of names in a complicated represented space will be a very complicated second-countable space.

Indeed, de Brecht et al. [10, Proposition 7.13] showed that $\mathcal{O}(\mathtt{Name}(\omega\langle k+1\rangle))$ does not (sequentially) embed into $\mathcal{O}(\mathtt{Name}(\omega\langle k\rangle))$ for any $k > 0$, where we use the symbol $\mathtt{Name}(X)$ to denote the set of all names of points in X for a represented space X; that is, $\mathtt{Name}(X) = \{\mathtt{Name}(x) : x \in X\}$. As $\mathtt{Name}(\omega\langle k\rangle)$ is a subspace of ω^ω, it is second-countable and T_0. We use this idea to show the following:

Theorem 2.29. *There is a represented cb_0 space X such that the $\mathcal{O}(X)$-degree are not covered by the $\mathcal{O}(\omega^\omega)$-degrees.*

Indeed, $X = \mathtt{Name}(\omega\langle 2\rangle)$ is such a space, and moreover, the collection of all $\mathcal{O}(\mathtt{Name}(\omega\langle 2\rangle))$-degrees is strictly bigger than that of all $\mathcal{O}(\omega^\omega)$-degrees.

Although the proof of de Brecht et al. [10, Proposition 7.13] does not work for proving Theorem 2.29, we make use of other embeddability results in [10]. They showed that $\omega\langle k+1\rangle$ sequentially embeds into $\mathcal{O}(\mathtt{Name}(\omega\langle k\rangle))$ (cf. [10, Theorem 7.4 and Corollary 7.10]). We give an effective direct proof of this fact.

Lemma 2.30. *For any $k \in \omega$, $\omega\langle k+1\rangle$ computably embeds into $\mathcal{O}(\mathtt{Name}(\omega\langle k\rangle))$.*

In particular, every $\omega\langle k+1\rangle$-degree is an $\mathcal{O}(\mathtt{Name}(\omega\langle k\rangle))$-degree.

Proof. First, there is a computable homeomorphism $i \colon \omega \times \omega\langle k\rangle \to \omega\langle k\rangle$. For any $F \in \omega\langle k+1\rangle$, consider the following set (which is roughly the set of all names of the graph of F):

$$U_F = \{p \in \omega^\omega : p \text{ is a name of } i(n, f) \text{ and } F(f) = n \text{ for some } (n, f) \in \omega \times \omega\langle k\rangle\}.$$

Clearly, U_F is an open subset of $\mathtt{Name}(\omega\langle k\rangle)$. It is not hard to see that $F \mapsto U_F$ is a computable embedding of $\omega\langle k+1\rangle$ into $\mathcal{O}(\mathtt{Name}(\omega\langle k\rangle))$ (cf. [10, Theorem 7.4]). $\square$

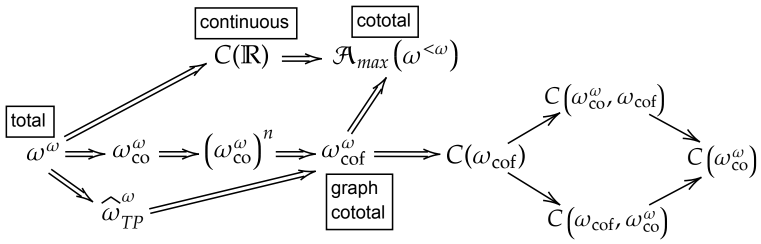

FIGURE 1. The relationship between degree structures

We shall use the following fact (essentially due to Dvornickov, cf. Normann [33, Corollary 7.2]); see also Theorem 3.21 below.

Fact 3 (see also Theorem 3.21)**.** *There is an $\omega\langle 3\rangle$-degree which is not an $\mathcal{O}(\omega^\omega)$-degree.*

Proof of Theorem 2.29. By Fact 3, we have an $\omega\langle 3\rangle$-degree $\mathbf{d}$ which is not an $\mathcal{O}(\omega^\omega)$-degree. Hence, by Lemma 2.30, such $\mathbf{d}$ is also an $\mathcal{O}(\mathtt{Name}(\omega\langle 2\rangle))$-degree.

We next check that every $\mathcal{O}(\omega^\omega)$-degree is an $\mathcal{O}(\mathtt{Name}(\omega\langle 2\rangle))$-degree. Hoyrup [14] showed that ω^ω is a computable retract of $C(\omega^\omega, 2)$. By the same argument, it is easy to see that ω^ω is a computable retract of $\omega\langle 2\rangle = C(\omega^\omega, \omega)$. Let (r, s) be a computable section-retraction pair witnessing this fact. Let $\delta\colon \mathtt{Name}(\omega\langle 2\rangle) \to \omega\langle 2\rangle$ be an admissible representation. Then, $s\colon \omega^\omega \to \omega\langle 2\rangle$ is tracked by a computable realizer $\tilde{s}\colon \omega^\omega \to \mathtt{Name}(\omega\langle 2\rangle)$, i.e., $s = \delta \circ \tilde{s}$. Now, it is easy to check that $(\tilde{s}, r \circ \delta)$ is a computable section-retraction pair witnessing that ω^ω is a computable retract of $\mathtt{Name}(\omega\langle 2\rangle)$. Hence, by Fact 1, $\mathcal{O}(\omega^\omega)$ is a computable retract of $\mathcal{O}(\mathtt{Name}(\omega\langle 2\rangle))$. This completes the proof. $\square$

3. Function spaces

We next examine the degree structures of function spaces with T_1 ranges. In the first half of this section, we deal with second countable spaces. The spaces depicted in the left half of Figure 1 are those dealt with in previous research [22], and this article will also deal with the spaces appearing in the right half of Figure 1. One of the main results of the first half of this section is that there is a $C(\omega_{\mathrm{cof}})$-degree which is not cototal (Theorem 3.6).

The last half of this section deals with the degree structures of higher type function spaces. One of the main results of the last half of this section is that the third order space $\omega\langle 2\rangle$ whose ground type is ω contains a functional whose degree is quasiminimal w.r.t. e-degrees (Theorem 3.13). Similarly, the third order space $\omega_{\mathrm{cof}}\langle 2\rangle$ whose ground type is ω_{cof} contains a functional which do not have an e-degree (Theorem 3.18).

3.1. **Lower topology:** $C(\mathbb{R}_<)$**.** Consider the function space on the lower space $\omega_<$. Here, a point in $\omega_<$ is a natural number, and a basic open set is of the form $[m, \infty] = \{n \in \omega : n \geq m\}$ for some $m \in \omega$. Note that a function $f\colon \omega_< \to \omega_<$ is continuous iff it is non-decreasing. In this case, we can consider the following network (see Section 1.3):

$$(\forall n)(\forall e)\ [e \leq f(n) \ \leftrightarrow\ (\exists d)\ [d \leq n \text{ and } \langle d, e\rangle \in N]].$$

However, it is clear that $\langle d, e\rangle \in N$ just indicates that $e \leq f(d)$.

Proposition 3.1. *The $C(\omega_<)$-degrees are exactly the e-degrees.*

Proof. Every $f \in C(\omega_<)$ has an e-degree since $E = \{\langle d, e\rangle : e \leq f(d)\}$ is a canonical local network at f. Conversely, let $A \subseteq \omega$ be given. Define $f(n) = 2n + 1$ if $n \in A$; otherwise $f(n) = 2n$. We claim that $\mathtt{Name}(f)$ is $\equiv_M$-equivalent to $\mathtt{Enum}(A)$. For $\leq_M$, before seeing $n \in A$, we enumerate $\langle n, i\rangle$ into N for any $i \leq 2n$, and if we see $n \in A$, enumerate $\langle n, 2n + 1\rangle \in N$. Conversely, given an enumeration of a local network N at f, if $\langle n, 2n + 1\rangle$ is enumerated into N, then enumerate $n \in A$. $\square$

Proposition 3.2. *The $C(\mathbb{R}_<)$-degrees are exactly the e-degrees.*

Proof. It is easy to see that the set of $\langle p, q\rangle$ such that $x > p$ implies $f(x) > q$ is a canonical local network at f. Other direction is verified by a similar argument as above. $\square$

3.2. **Telophase topology: $C(\hat{\omega}_{TP})$.** Roughly speaking, the telophase space $\hat{\omega}_{TP}$ looks like a "two-point compactification" of ω. This is the set $\omega \cup \{\infty, \infty_\star\}$ endowed with the topology generated by $\{\{m\}, [m, \infty], [m, \infty_\star] : m \in \omega\}$, where $[m, \infty] = \{n \in \omega : n \geq m\} \cup \{\infty\}$ and $[m, \infty_\star] = \{n \in \omega : n \geq m\} \cup \{\infty_\star\}$. The $(\hat{\omega}_{TP})^\omega$-degrees have been studied in [22].

Proposition 3.3. *The $C(\hat{\omega}_{TP})$-degrees are exactly the $(\hat{\omega}_{TP})^\omega$-degrees.*

Proof. For $f \in C(\hat{\omega}_{TP})$, the restriction $f \upharpoonright \omega$ of f up to ω is exactly an element of $(\hat{\omega}_{TP})^\omega$. Except for this $(\hat{\omega}_{TP})^\omega$-information, f only has two values $f(\infty)$ and $f(\infty_\star)$. Note that every point in $\hat{\omega}_{TP}$ is computable, so the information on these two computable values does not affect the degree of f; that is, $f \upharpoonright \omega \equiv_T f$, where the former is in $(\hat{\omega}_{TP})^\omega$. $\square$

3.3. **Cofinite topology: $C(\omega_{\mathrm{cof}})$.** We now consider the function space on the cofinite space ω_{cof}. Note that a function $f\colon \omega_{\mathrm{cof}} \to \omega_{\mathrm{cof}}$ is continuous iff it is finite-to-one or constant. This is because any singleton $\{c\}$ is a closed set in ω_{cof}, so if f is continuous, $f^{-1}\{c\}$ is also closed, which is finite or ω_{cof}. In this space, we can consider the following network (see Section 1.3):

$$(\forall n)(\forall e)\ [f(n) \neq e \ \leftrightarrow\ (\exists D)\ [n \notin D \text{ and } \langle D, e\rangle \in N]]. \tag{5}$$

Proposition 3.4. *Every $C(\omega_{\mathrm{cof}})$-degree has an e-degree. Indeed, the following set E_f is a canonical local network at f:*

$$E_f = \{\langle D, e\rangle : f^{-1}\{e\} \subseteq D\}.$$

Proof. We first claim that E_f is a local network at f; indeed, E_f satisfies the condition (5). If $f(n) = e$, then $n \in f^{-1}\{e\}$. Therefore, for any D, if $\langle D, e\rangle \in E_f$ then $f^{-1}\{e\} \subseteq D$, so $n \in D$. This means that if $n \notin D$ implies $\langle D, e\rangle \notin E_f$. If $f(n) \neq e$, then $n \notin f^{-1}\{e\}$, and therefore, there is D such that $n \notin D$ and $f^{-1}\{e\} \subseteq D$, which means $\langle D, e\rangle \in E_f$.

We next show that E_f is canonical. Let N be a local network at f. If we see $\langle D_i, e\rangle \in N$ for finitely many i, then enumerate all $\langle C, e\rangle$ into E such that $\bigcap_i D_i \subseteq C$. It is clear that $E \leq_e N$, and this procedure is independent of the choice of N. We then claim that $E = E_f$.

To see $E \subseteq E_f$, assume that $\langle C, e\rangle$ is enumerated into E. In this case, $\langle D_i, e\rangle$ for finitely many i are enumerated into N, and $\bigcap_i D_i \subseteq C$. If $n \in f^{-1}\{e\}$, then by the definition of a local network, $\langle D_i, e\rangle \in N$ implies $n \in D_i$, which also implies that $n \in C$. Therefore, $f^{-1}\{e\} \subseteq C$, and thus $\langle C, e\rangle \in E_f$.

To see $E_f \subseteq E$, we first note that for any $i \notin f^{-1}\{e\}$, there is D_i such that $i \notin D_i$ and $\langle D_i, e\rangle \in N$. Choose such D_i for any $i \notin f^{-1}\{e\}$, and pick $j \notin f^{-1}\{e\}$. Let $\{k_j\}_{j<s}$ be an enumeration of $D_j \setminus f^{-1}\{e\}$. Since $\langle D_i, e\rangle \in N$ implies $f^{-1}\{e\} \subseteq D_i$, we clearly have $f^{-1}\{e\} = D_j \cap \bigcap_{t<s} D_{k_t}$. Thus, by our definition of E, for any D with $f^{-1}\{e\} \subseteq D$, $\langle D, e\rangle$ is enumerated into E. $\square$

The above shows that the space $C(\omega_{\rm cof})$ is second-countable via the subbasis $B_{D,e} = \{f : f^{-1}\{e\} \subseteq D\}$.

Remark. The space $C(\omega_{\rm cof})$ is not T_2 since any two open sets have an intersection. To see this, given $B_{D,d}$ and $B_{E,e}$, consider a function f such that $f^{-1}\{d, e\} = \emptyset$.

Let us compare the degree structures of $\omega^\omega_{\rm cof}$ and $C(\omega_{\rm cof})$. First, we can confirm that the latter is larger.

Proposition 3.5. *Every $\omega^\omega_{\rm cof}$-degree is a $C(\omega_{\rm cof})$-degree.*

Proof. Given $g\colon \omega \to \omega$, define $f(n) = \langle n, g(n)\rangle$. Clearly, $f^{-1}\{\langle n, k\rangle\} \subseteq \{n\}$. Thus, we start enumerating all $\langle d, \langle n, k\rangle\rangle$ with $\{n\} \subseteq D_d$. If we see $g(n) \neq k$, we enumerate $\langle d, \langle n, k\rangle\rangle$ for all d. Hence, a canonical local network E_f of f is e-reducible to the co-graph of g. Conversely, if $\langle d, \langle n, k\rangle\rangle$ with $n \notin D_d$ is enumerated into a network E of f, then declare $g(n) \neq k$. $\square$

Recall that every graph-cototal degree is cototal [1]; that is, every $\omega^\omega_{\rm cof}$-degree is an $\mathcal{A}_{\max}(\omega^{<\omega})$-degree. The following shows that the $C(\omega_{\rm cof})$-degrees are quite large, which do not included in the cototal degrees.

Theorem 3.6. *There exists a $\emptyset''$-computable $C(\omega_{\rm cof})$-degree which is not cototal.*

Proof. We shall need to construct a function $f\colon \omega \to \omega$ (thought of as a point in $C(\omega_{\rm cof})$) satisfying the requirements

$$\begin{aligned} R_e\colon \mathtt{Nbase}(f) = \Phi_e\left(\Psi_e\left(\mathtt{Nbase}(f)\right)\right) \\ \Rightarrow \omega^{<\omega} \setminus \Psi_e\left(\mathtt{Nbase}(f)\right) \text{ is not a maximal antichain in } \omega^{<\omega}. \end{aligned}$$

Here recall $\mathtt{Nbase}(f) = \{\langle n, D\rangle\colon D \text{ is a finite set such that } D \supseteq f^{-1}\{n\}\}$ and we are identifying each $\Psi_e\left(\mathtt{Nbase}(f)\right)$ as a subset of $\omega^{<\omega}$. This suffices to verify the assertion since the $\mathcal{A}_{\max}(\omega^{<\omega})$-degrees are exactly the cototal degrees, where note that the space $\mathcal{A}_{\max}(\omega^{<\omega})$ can be thought of as a subspace of $\mathcal{O}(\omega^{<\omega})$ consisting of the complements of maximal antichains in $\omega^{<\omega}$.

We shall construct a $\emptyset''$-enumeration of $\mathtt{Nbase}(f)$. For each stage s, we determine a parameter n_{s+1} and a sequence $\mathcal{L}_s = (L_m)_{m<n_{s+1}}$ of finite subsets of ω, where $n_0 = 0$ and $\mathcal{L}_0$ is the empty sequence. This is a list of declarations $f^{-1}\{m\} = L_m$ for any $m < n_{s+1}$. In other words, we only enumerate pairs $\langle m, D\rangle$ with $L_m \subseteq D$ into the set $\mathtt{Nbase}(f)$. In order to make f a total function, $(L_m)_{m\in\omega}$ must eventually be a partition of ω.

For a finite list $\mathcal{L} = (L_m)_{m\in I}$, we say that a set $A \subseteq \omega$ is *$\mathcal{L}$-consistent* if, for any $m \in I$, $\langle m, D\rangle \in A$ implies $L_m \subseteq D$. Note that an $\mathcal{L}$-consistent finite set A determines a finite list $\mathcal{L}_A = (L_m)_{m\in I\cup J}$ extending $\mathcal{L}$, where J is a finite set such that $I \cap J = \emptyset$. To be more explicit, if $m \notin I$ and $\langle m, D\rangle \in A$ for some D, then define $L_m = \bigcap\{D : \langle m, D\rangle \in A\}$. Clearly, if $A \subseteq \mathtt{Nbase}(f)$ then $f^{-1}\{m\} \subseteq L_m$.

At stage $s = 2e$, we deal with enumeration operators $\Phi = \Phi_e$ and $\Psi = \Psi_e$. The following is the R_s-strategy:

Step 1. Let L_{n_s} be the set of all $k < s$ such that $k \notin L_m$ for any $m < n_s$. This action guarantees that the union of L_m's will eventually become ω.

Step 2. Put $n = n_s + 1$. In a $\emptyset'$-computable manner, check whether there exist a finite set $E \subseteq \omega^{<\omega}$ and an $(\mathcal{L}_s * L_{n_s})$-consistent finite set A satisfying the following condition:

$$\langle n, \emptyset \rangle \in \Psi(E) \text{ and } E \subseteq \Phi(A).$$

Case 2a. If such E and A exist, let J_A be the set of all values mentioned by A; that is, $J_A = \{m \in \omega : \langle m, D \rangle \in A \text{ for some } D\}$. Then let $\mathcal{L}'$ be the list obtained by adding the declarations $L_m = \emptyset$ for any $m \in J_A$ with $m > n$ to $(\mathcal{L}_s * L_{n_s})$. Note that $\mathcal{L}'$ does not contain any declaration concerning $f^{-1}\{n\}$. Then go to Step 3 below.

Verification. Assume that f follows the declarations by $\mathcal{L}'$, and $\Phi(\Psi(\mathtt{Nbase}(f))) = \mathtt{Nbase}(f)$ holds. If the strategy reaches Case 2a, we claim that $f^{-1}\{n\} = \emptyset$ if and only if $E \subseteq \Phi(\mathtt{Nbase}(f))$. If $f^{-1}\{n\} = \emptyset$ then as f follows $\mathcal{L}'$, we must have $A \subseteq \mathtt{Nbase}(f)$. This is because if $\langle m, D \rangle \in A$ then $L_m = \emptyset$, which forces $f^{-1}\{m\} = \emptyset$, so $\langle m, C \rangle$ is enumerated into $\mathtt{Nbase}(f)$ for any finite set C; hence $\langle m, D \rangle \in \mathtt{Nbase}(f)$. By monotonicity, $A \subseteq \mathtt{Nbase}(f)$ implies $\Phi(A) \subseteq \Phi(\mathtt{Nbase}(f))$, so we get $E \subseteq \Phi(\mathtt{Nbase}(f))$ by our construction. Conversely, if $E \subseteq \Phi(\mathtt{Nbase}(f))$ then $\langle n, \emptyset \rangle \in \Psi(E) \subseteq \Psi(\Phi(\mathtt{Nbase}(f)))$. Thus, if $\Psi(\Phi(\mathtt{Nbase}(f))) = \mathtt{Nbase}(f)$ then we must have $f^{-1}\{n\} = \emptyset$.

Case 2b. If no such E and A exist, add the new declaration $L_n = \emptyset$, which forces $f^{-1}\{n\} = \emptyset$. Put $n_{s+1} = n_s + 2$ and $\mathcal{L}_{s+1} = \mathcal{L}_s * L_{n_s} * L_n$. Put $I_{s+1} = I_s$. Then go to the next stage $s + 1$.

Verification. Assume that f follows the declarations by $\mathcal{L}_{s+1}$. If the strategy reaches Case 2b, as f follows $\mathcal{L}_{s+1}$, any finite set $A \subseteq \mathtt{Nbase}(f)$ is $(\mathcal{L}_s * L_{n_s})$-consistent. Reaching Case 2b means $\langle n, \emptyset \rangle \notin \Psi(E)$ for any finite set $E \subseteq \Phi(A)$. Hence, we get $\langle n, \emptyset \rangle \notin \Psi(\Phi(\mathtt{Nbase}(f)))$. This means that if $\mathtt{Nbase}(g) = \Psi(\Phi(\mathtt{Nbase}(f)))$ then $g^{-1}\{n\} \neq \emptyset$. However, if f follows $\mathcal{L}_{s+1}$ then we must have $f^{-1}\{n\} = \emptyset$. Hence, $\mathtt{Nbase}(f) \neq \Psi(\Phi(\mathtt{Nbase}(f)))$, which fulfills the requirement R_e.

Step 3. Let $\mathcal{L}'_0$ be the list obtained by adding the declaration $L_n = \emptyset$ to $\mathcal{L}'$, which forces $f^{-1}\{n\} = \emptyset$. Enumerate E (chosen in Case 2a) as $\{\alpha_i\}_{i<k}$, and then construct an increasing sequence $\mathcal{L}'_0 \subseteq \mathcal{L}'_1 \subseteq \cdots \subseteq \mathcal{L}'_k$ of lists of declarations.

At each round $i < k$, in a $\emptyset''$-computable manner, ask whether there exist a string $\beta \in \omega^{<\omega}$ which is comparable with α_i and an $\mathcal{L}'_i$-consistent finite set B such that $\beta \notin \Phi(C)$ for any $\mathcal{L}_B$-consistent finite set C, where $\mathcal{L}_B$ is the finite list extending $\mathcal{L}'_i$ determined by B as mentioned above. If such β and B exist, put $\mathcal{L}'_{i+1} = \mathcal{L}_B$. If $i + 1 < k$, go to round $i + 1$.

Case 3a. If such β and B exist for any $i < k$. Then let $\mathcal{L}''$ be the list obtained by removing the declaration $L_n = \emptyset$ from $\mathcal{L}'_k$, and instead adding the declaration $L_n = \{a\}$, where a is a large fresh number, so $\mathcal{L}''$ declares $f^{-1}\{n\} = \{a\}$.

Let n_{s+1} be the least number such that the mth term L_m in $\mathcal{L}''$ is undefined for any $m \geq n_{s+1}$. For each $m < n_{s+1}$, if L_m is undefined, then declare $L_m = \emptyset$. Let

$\mathcal{L}_{s+1}$ be the resulting list $(L_m)_{m<n_{s+1}}$. Put $I_{s+1} = I_s$. Then go to the next stage $s+1$.

Verification. Assume that f follows the declarations by $\mathcal{L}_{s+1}$, and $\Phi(\Psi(\mathtt{Nbase}(f))) = \mathtt{Nbase}(f)$ holds. In particular, f follows $\mathcal{L}'$, so as mentioned above, when we reach Case 2a, it is ensured that $f^{-1}\{n\} = \emptyset$ if and only if $E \subseteq \Phi(\mathtt{Nbase}(f))$. However, f follows $\mathcal{L}_{s+1}$, which forces $f^{-1}\{n\} = \{a\}$, so we have $E \not\subseteq \Phi(\mathtt{Nbase}(f))$. Thus, there must exist $\alpha_i \in E$ such that $\alpha_i \notin \Phi(\mathtt{Nbase}(f))$. Let β be a string chosen in Case 3a. If $\beta \in \Phi(\mathtt{Nbase}(f))$ then there is a finite set $C \subseteq \mathtt{Nbase}(f)$ such that $\beta \in \Phi(C)$. As f follows $\mathcal{L}_{s+1}$, any finite set $C \subseteq \mathtt{Nbase}(f)$ is $\mathcal{L}_{s+1}$-consistent, and in particular, $\mathcal{L}''$-consistent since $\mathcal{L}''$ is a sublist of $\mathcal{L}_{s+1}$. The only difference between $\mathcal{L}''$ and $\mathcal{L}'_k$ is whether the nth term is $\{a\}$ or $\emptyset$; however, $\{a\} \subseteq D$ implies $\emptyset \subseteq D$, so $\mathcal{L}''$-consistency implies $\mathcal{L}'_k$-consistency. Moreover, since $\mathcal{L}'_{i+1}$ is a sublist of $\mathcal{L}'_k$, any finite $C \subseteq \mathtt{Nbase}(f)$ is $\mathcal{L}'_{i+1}$-consistent. However, by our assumption, we must have $\beta \notin \Phi(C)$, which is impossible. Consequently, we obtain $\alpha_i, \beta \notin \Phi(\mathtt{Nbase}(f))$. This means that the complement of $\Phi(\mathtt{Nbase}(f))$ contains two comparable strings α_i and β, which means that this is not an antichain. Thus the requirement R_e is fulfilled.

Case 3b. If no such pair (β, B) exist for some i, let $i(e)$ be the least such i, and put $\alpha_e^\star = \alpha_{i(e)}$. Let n_{s+1} be the least number such that the mth term L_m in $\mathcal{L}_e^\star = \mathcal{L}'_{i(e)}$ is undefined for any $m \geq n_{s+1}$. For each $m < n_{s+1}$, if L_m is undefined, then declare $L_m = \emptyset$. Let $\mathcal{L}_{s+1}$ be the resulting list $(L_m)_{m<n_{s+1}}$. Put $I_{s+1} = I_s \cup \{e\}$. Then go to the next stage $s+1$.

At stage $s = 2e+1$, we continue the R_e-strategy for each $e \in I_s$. Note that I_s is the list of all e's such that the R_e-strategy reaches Case 3b. In this case, for any β comparable with $\alpha_e^\star$ and $\mathcal{L}_e^\star$-consistent finite set B, there exists an $\mathcal{L}_B$-consistent finite set C such that $\beta \in \Phi(C)$.

Let B be the current approximation of $\mathtt{Nbase}(f)$, which must be $\mathcal{L}_s$-consistent. If necessary, we can guarantee that B determines $\mathcal{L}_s$ by slightly extending B. To be more explicit, add $\langle m, L_m\rangle$ to B for each $m < n_s$. Also, since $\mathcal{L}_e^\star$ is a sublist of $\mathcal{L}_s$, B is $\mathcal{L}_e^\star$-consistent, and by our convention, $\mathcal{L}_B$ extends $\mathcal{L}_s$. Therefore, by our assumption, for any β comparable with $\alpha_e^\star$, there exists an $\mathcal{L}_s$-consistent set C that satisfies $\beta \in \Phi(C)$.

Continue this procedure to extend the current approximation of $\mathtt{Nbase}(f)$ to an $\mathcal{L}_s$-consistent set C such that for each $e \in I_s$, the first s strings that are comparable to $\alpha_e^\star$ are in $\Phi(C)$. Let n_{s+1} be a fresh value larger than any value mentioned by C, and $\mathcal{L}_{s+1} = (L_m)_{m<n_{s+1}}$ be a suitable extension of $\mathcal{L}_s$ such that $\mathcal{L}_C$-consistency implies $\mathcal{L}_{s+1}$-consistency. Put $I_{s+1} = I_s$. Then go to the next stage $s+1$.

Verification. Assume that f follows the declarations by $\mathcal{L}_{s+1}$, and $\Phi(\Psi(\mathtt{Nbase}(f))) = \mathtt{Nbase}(f)$ holds. Recall that the nth term of $\mathcal{L}_e^\star$ is $\emptyset$, so this ensures $f^{-1}\{n\} = \emptyset$ since f follows $\mathcal{L}_{s+1}$ which extends $\mathcal{L}_e^\star$. As mentioned above, when we reach Case 2a, it is ensured that $f^{-1}\{n\} = \emptyset$ if and only if $E \subseteq \Phi(\mathtt{Nbase}(f))$. Since $\alpha_e^\star \in E$, we get $\alpha_e^\star \in \Phi(\mathtt{Nbase}(f))$. Now, by our construction, any string β comparable with $\alpha_e^\star$ is eventually enumerated into $\Phi(\mathtt{Nbase}(f))$. This means that the complement of $\Phi(\mathtt{Nbase}(f))$ is not maximal. Thus the requirement R_e is fulfilled. $\square$

Remark. Let us comment on the topological properties of the space $C(\omega_{\mathrm{cof}})$. The space $C(\omega_{\mathrm{cof}})$ is a $G_{\delta\sigma}$-space, that is, every closed set is $G_{\delta\sigma}$. To see this, note that

$[i;n] := \{f : f(i) = n\}$ is closed in $C(\omega_{\rm cof})$, and therefore, $\{f : f(i) \neq n\}$ is F_σ since it is of the form $\bigcup_{m\neq n}[i;m]$. Hence, every basic open set $B_{D,n} = \{f : f^{-1}\{n\} \subseteq D\}$ can be written as $\bigcap_{d\notin D}\bigcup_{m\neq n}[i;m]$, which is $F_{\sigma\delta}$. However the relativization of Theorem 3.6 shows that $C(\omega_{\rm cof})$ is not a G_δ-space (since cototal degrees are degrees of points in computably G_δ-spaces [22]).

3.4. **Cofinite topology restricted to bounded functions: $C_b(\omega_{\rm cof})$.** Let $C_b(\omega_{\rm cof})$ be the subspace of $C(\omega_{\rm cof})$ consisting of computably bounded functions, that is, the set of all g such that $g(n) < b(n)$ for some computable function $b \in \omega^\omega$.

Proposition 3.7. *The $C_b(\omega_{\rm cof})$-degrees are exactly the total enumeration degrees.*

Proof. Let φ be a total computable function and $X = \mathtt{Nbase}_{C_b(\omega_{\rm cof})}(f)$ for some $f \in \omega^\omega$ where $f(x) \leq \varphi(x)$ for every x. Define A by the following. For each finite set D and each $i \in \omega$, we let $\langle D, i\rangle \in A$ if and only if there exists some finite set E and some $k \in E \setminus D$ such that $\langle E, i\rangle \in X$, and for every $j \neq i$ and $j \leq \varphi(k)$, there exists some finite D_j such that $k \notin D_j$ and $\langle D_j, j\rangle \in X$. Clearly, $A \leq_e X$. We claim that $A = X^c$. Fix a D and i. If $\langle D, i\rangle \in X \cap A$, then $f(k) \neq i$ as $\langle D, i\rangle \in X$. However $f(k)$ must be $\leq \varphi(k)$, but then $\langle D_{f(k)}, f(k)\rangle \in X$, which is impossible. On the other hand, if $\langle D, i\rangle \notin X$, then $f^{-1}\{i\}$ must contain some $k \notin D$. Hence $\langle D, i\rangle \in A$, where we can take $E = f^{-1}\{i\}$. This shows that $A = X^c$, and hence $X^c \leq_e X$.

Conversely, given $A \subseteq \omega$, consider the function $f(n) = 2n$ if $n \in A$ and $f(n) = 2n + 1$ if $n \notin A$. □

In view of Proposition 3.7 we will consider a more general notion of being "computably bounded". Let $C_A(\omega_{\rm cof})$ be the subspace of $C(\omega_{\rm cof})$ consisting of A-computably bounded functions, that is, the set of all g such that $g(n) < b(n)$ for some A-computable function $b \in \omega^\omega$. Similarly, let $(\omega^\omega_{\rm co})_A$ ($C_A(\omega, \omega_{\rm cof})$, resp.) be the subspace of $\omega^\omega_{\rm co}$ ($C(\omega, \omega_{\rm cof})$, resp.) consisting of all A-computably bounded functions.

Observation 3.8. *For any A, $(\omega^\omega_{\rm co})_A$ computably embeds into $C_A(\omega, \omega_{\rm cof})$, and $C_A(\omega, \omega_{\rm cof})$ computably embeds into $C_A(\omega_{\rm cof})$. In particular, there is a quasi-minimal $C_{\emptyset'}(\omega_{\rm cof})$-degree.*

Proof. Two embeddings are given by $x \mapsto \lambda n.x \upharpoonright n$ and $g \mapsto \lambda n.\langle n, g(n)\rangle$. See also Proposition 3.5. For the second assertion, it is not hard to see that there is a $\emptyset'$-computably bounded $x \in \omega^\omega_{\rm co}$ which is quasi-minimal, by using the property that every two open sets in $\omega^\omega_{\rm co}$ intersect; see [22]. □

Proposition 3.9. *For any A, there is a continuous degree which is not a $C_A(\omega_{\rm cof})$-degree. There is a $C_{\emptyset'}(\omega_{\rm cof})$-degree which is not a continuous degree.*

Proof. For any $g \in C_A(\omega_{\rm cof})$, one can show that $g \oplus A$ is total as in the proof of Proposition 3.7. This means that for any $g \in C_A(\omega_{\rm cof})$, if the lower cone $\{x \in \omega^\omega : x \leq_T g\}$ contains A then it is a principal Turing ideal. However, as proven by Miller [31], every countable Scott ideal (i.e., ω-model of weak König's lemma) is of the form $\{x \in \omega^\omega : x \leq_T f\}$ for some $f \in [0, 1]^\omega$, and a Scott ideal is not principal. This shows that there is a continuous degree which is not a $C_A(\omega_{\rm cof})$-degree.

For the second assertion, there is no quasi-minimal continuous degree (see Miller [31]) while there is a quasi-minimal $C_{\emptyset'}(\omega_{\rm cof})$-degree by Observation 3.8. □

Proposition 3.10. *For any A, there exists an $(\omega^\omega_{\rm co})_{A'}$-degree which is not a $C_A(\omega_{\rm cof})$-degree.*

Proof. For any X, we construct an A'-computably bounded $g \in \omega^\omega_{\rm co}$ which is not X-computable, and for any $h \in C_A(\omega_{\rm cof})$, if $h \leq_T g$ then h is A-computable. At stage $s = (d, e)$, we ensure that if $\Phi_e(g) = h \in C_A(\omega_{\rm cof})$ and h is bounded by φ^A_d then h is computable.

Let $b \leq_T A'$ be a function dominating all partial A-computable functions. There are two cases. First, assume that there exist infinitely many n such that for any $i < \varphi^A_d(n)\downarrow$, there is E^n_i such that E^n_i is consistent with g_s, and $\langle D, i\rangle$ is enumerated into $\Phi_e(E^n_i)$ for some $D \not\ni n$. In this case, let $u(n)$ be the largest number mentioned in $(E^n_i)_{i<\varphi^A_d(n)}$. Since u is a partial A-computable function, we have $u(n) < b(n)$ for almost all n. So let n be such that $u(n)\downarrow$ and $u(n) < b(n)$. Then, there is a b-bounded finite string g^*_s extending g_s consistent with $(E_i)_{i<\varphi^A_d(n)}$. This ensures that for any $i < \varphi^A_d(n)$, there is $D \not\ni n$ such that $\langle D, i\rangle \in \Phi_e(g)$, so, if $h = \Phi_e(g)$ then $h^{-1}\{i\} \subseteq D \not\ni n$, which forces $h(n) \geq \varphi^A_d(n)$.

If not, for almost all n, either $\varphi^A_d(n)\uparrow$ or there is $i < \varphi^A_d(n)$ such that if E is consistent with g_s and $\langle D, i\rangle \in \Phi_e(E)$ then $n \in D$. In this case, put $g^*_s = g_s$. Assume that $\Phi_e(g)$ defines a function. Then note that we must have $i = h(n)$. This is because we have that $h^{-1}\{i\} \subseteq D$ imples $n \in D$ by assumption, which implies that $h(n) = i$.

Assume that φ^A_d is total, and $h = \Phi_e(g)$ is bounded by φ^A_d. Given n, if $j \neq h(n)$, then there must exist (D, E) such that $\langle D, j\rangle \in \Phi_e(E)$ and $n \notin D$. As in the proof of Proposition 3.7, it is not hard to show that h is A-computable.

We now want to ensure that g is not X-computable. Check if a sufficiently long b-bounded string σ extending g^*_s is enumerated into the s-th X-c.e. set $W_s(X)$, that is, if $W_s(X)$ computes an element $g \in \omega^\omega_{\rm co}$ then $g \notin [\sigma]$. If such σ exists, then put $g_{s+1} = \sigma$. If such σ does not exist, then put $g_{s+1} = g^*_s$. Here note that if $W_s(X)$ computes an element of $\omega^\omega_{\rm co}$, then it must enumerate an arbitrarily long b-bounded string extending any given b-bounded finite string. Therefore, it is not hard to verify that this strategy ensures that g is not X-computable. □

3.5. **Cocylinder topology:** $C(\omega^\omega_{\rm co}, Y)$**.** Now consider the cocylinder space $\omega^\omega_{\rm co}$. A function $f\colon \omega^\omega_{\rm co} \to \omega^\omega_{\rm co}$ is continuous iff $f^{-1}[\sigma]$ is finitely generated (i.e., it is a finite union of cylinders) for any $\sigma \in \omega^{<\omega}$. In particular, f is continuous w.r.t. the standard Baire topology on ω^ω. Hence, $C(\omega^\omega_{\rm co})$ is included in $C(\omega^\omega)$ as a set.

In this space, we can consider the following network:

$$(\forall x)(\forall\sigma)\ [f(x) \not\succ \sigma \ \leftrightarrow\ (\exists D)\ [x \notin [D] \text{ and } \langle D, \sigma\rangle \in N]],$$

where $[D]$ is the set of all $x \in \omega^\omega$ extending some $\sigma \in D$.

Corollary 3.11. *Every $C(\omega_{\rm cof})$-degree is a $C(\omega_{\rm cof}, \omega^\omega_{\rm co})$-degree, and every $C(\omega_{\rm cof}, \omega^\omega_{\rm co})$-degree is a $C(\omega^\omega_{\rm co})$-degree.*

Proof. One can see that the map $n^\frown 0^\omega \mapsto n$ gives a computable retraction from $\omega^\omega_{\rm co}$ to $\omega_{\rm cof}$. Hence, $\omega_{\rm cof}$ is a computable retract of $\omega^\omega_{\rm co}$. By Fact 1, $C(\omega_{\rm cof})$ is a computable retract of $C(\omega_{\rm cof}, \omega^\omega_{\rm co})$ and $C(\omega^\omega_{\rm co}, \omega_{\rm cof})$, and moreover, the latter two spaces are computable retracts of $C(\omega^\omega_{\rm co})$. This implies the assertion by Observation 2.17. □

Proposition 3.12. *The following set E_f is a canonical local network at $f \in C(\omega^\omega_{\rm co}, \omega_{\rm cof})$:*

$$E_f = \{\langle D, e\rangle : f^{-1}\{e\} \subseteq [D]\}.$$

Similarly, the following set E_f is a canonical local network at $f \in C(\omega^\omega_{\rm co})$:

$$E_f = \{\langle D, \sigma\rangle : f^{-1}[\sigma] \subseteq [D]\}.$$

Hence, every $C(\omega^\omega_{\rm co})$-degree has an e-degree.

Proof. We only show the first assertion. We first claim that E_f is a local network at f. If $f(x) = e$, then $x \in f^{-1}\{e\}$. Therefore, for any D, if $\langle D, e\rangle \in E_f$ then $x \in [D]$. If $f(x) \neq e$, then by continuity of f, $\omega^\omega \setminus f^{-1}\{e\} = f^{-1}[\omega \setminus \{e\}]$ is an open set containing x, and therefore, there is an open neighborhood U of x such that $x \subseteq U \subseteq \omega^\omega \setminus f^{-1}\{e\}$. There is such an U of the form $\omega^\omega \setminus [D]$ for some finite set D of strings. Then, $x \notin [D]$ and $f^{-1}\{e\} \subseteq [D]$, which means $\langle D, e\rangle \in E_f$.

We next show that E_f is canonical. Let N be a local network of f. If we see $\langle D_i, e\rangle \in N$ for finitely many i, then enumerate all $\langle C, e\rangle$ into E such that $\bigcap_i [D_i] \subseteq [C]$. It is clear that $E \leq_e N$, and this procedure is independent of the choice of N. We then claim that $E = E_f$.

To see $E \subseteq E_f$, assume that $\langle C, e\rangle$ is enumerated into E. In this case, $\langle D_i, e\rangle$ for finitely many i are enumerated into N, and $\bigcap_i [D_i] \subseteq [C]$. If $x \in f^{-1}\{e\}$, then by the definition of a local network, $\langle D_i, e\rangle \in N$ implies $x \in [D_i]$, which also implies that $x \in C$. Therefore, $f^{-1}\{e\} \subseteq C$, and thus $\langle C, e\rangle \in E_f$.

To see $E_f \subseteq E$, assume that $f^{-1}\{e\} \subseteq [D]$. We first claim that there is C such that $\langle C \cup D, e\rangle \in E$, and every $\sigma \in C$ is incomparable with any string in D. Otherwise, $\langle D', e\rangle \in N$ implies that D' always contains a proper initial segment of a string in D. Since there are only finitely many initial segments of strings in D, there is $\sigma \in D$ such that D' always contains some $\tau \prec \sigma$. However, as N is a network, it is impossible.

Now, fix such a C. We next claim that for any $\sigma \in C$, there is F such that $\langle F, e\rangle \in E$ and $[F] \cap [\sigma] = \emptyset$. Otherwise, fix an enumeration $(F_n)_{n\in\omega}$ of all finite sets such that $\langle F_n, e\rangle \in E$. Define $G_n = \bigcap_{i\leq n} F_i$, and then $(G_n)_{n\in\omega}$ is a decreasing sequence. Since $\langle G_n, e\rangle \in E$, by our assumption, $[G_n] \cap [\sigma] \neq \emptyset$. Consider $H_n = \{\tau \in G_n : \tau \succeq \sigma\}$. It is not hard to see that every $\tau \in H_{n+1}$ extends some string in H_n. Therefore, the downward closure of $\bigcup_n H_n$ defines a finite branching tree T. By König's lemma, there is an infinite path x through T. Since x extends a string in H_n, we have $x \in [G_n]$. As $x \succ \sigma \in C$, by our choice of C, we have $x \notin [D]$, and in particular, $x \notin f^{-1}\{e\}$. By our definition of G_n, there is no G such that $x \notin [G]$ and $\langle G, \sigma\rangle \in N$. A contradiction.

Consequently, for any $\sigma \in C$, there is F such that $\langle F, e\rangle \in E$ and $[F] \cap [\sigma] = \emptyset$. As C is finite, there exists G such that $\langle G, e\rangle \in E$ and $[G] \cap [C] = \emptyset$. By our choice of C, this concludes that $\langle D, e\rangle \in E$.

The same argument applies to the second assertion. □

Note that the $C(\omega^\omega_{\rm co})$-degrees are strictly smaller than the e-degrees, since $C(\omega^\omega_{\rm co})$ is T_1 (Section 1.3), and there is a T_1-quasi-minimal e-degree [22].

3.6. **Third order Baire space: $C(\omega^\omega, Y)$.** Let us look at the degree structures of higher order function spaces. Hinman [13] showed that the degree structure of $C(\omega^\omega, \omega)$ is strictly larger than the total degrees. Kihara-Pauly [24] showed that the degree structure of $\mathcal{O}(\omega^\omega)$ is strictly larger than the e-degrees. Here we show a much stronger result: Roughly speaking, there is a type 2 functional which has no nontrivial "second countable information".

Theorem 3.13. *There exists a $C(\omega^\omega, \omega)$-degree which is quasiminimal with respect to all e-degrees.*

Proof. We will define $f \in C(\omega^\omega, \omega)$ such that $\mathtt{Name}(f)$ contains no computable member, and such that for every set $A \subseteq \omega$, if $\mathtt{Enum}(A) \leq \mathtt{Name}(f)$ then A is c.e. We wish to satisfy the following requirements:

$$P_e : \text{If } \varphi_e \text{ is total} \Rightarrow \varphi_e \notin \mathtt{Name}(f).$$

$$R_e : \text{If } \Phi_e\left(\mathtt{Name}(f)\right) \subseteq \mathtt{Enum}(A) \text{ for some } A \Rightarrow A \text{ is c.e.}$$

The construction is a straightforward infinite injury argument. Our function f shall have a very simple definition; at the end, we will build f such that for every n, either $f^{-1}\{2n\} = [n]$ and $f^{-1}\{2n+1\} = \emptyset$, or $f^{-1}\{2n\} = [n] - [\tau]$ and $f^{-1}\{2n+1\} = [\tau]$ for some $\tau \supset [n]$. (We can also make f $\{0,1\}$-valued, but we choose the range of f to be infinite to make notation simpler).

We define a computable approximation f_s to f. At every stage s, and every n, we let $f_s(n) = 0$ represent the first alternative, and $f_s(n) = \tau$ represent the second alternative. Our approximation to f will initially start off with $f_s(n) = 0$ and we will make at most one change to $f_s(n)$ for each n. For each s, we let α_s be the "canonical" element of $\mathtt{Name}(f_s)$ obtained by enumerating, for each n, $\langle \sigma, n\rangle$ for the set of all minimal strings σ such that $[\sigma] \subseteq f_s^{-1}\{n\}$. Another parameter which we shall need during the construction is $\beta_s(\tau_0)$, where $\tau_0 \supset k_0$, and k_0 is an integer such that $f_s(k_0) = 0$. This is the "slowed down" element of $\mathtt{Name}(f_s)$ defined the following way. $\beta_s(\tau_0)$ enumerates all elements of α_s which are incomparable with $[k_0]$. It also enumerates, for each $j \in \omega$, the element $\langle \tau_0{}^\frown j, 2k_0\rangle$. Finally, it also enumerates $\langle \sigma, 2k_0\rangle$ for the set of all minimal strings $\sigma \supset [k_0]$ which are incomparable with τ_0. Obviously, $\beta_s(\tau_0) \in \mathtt{Name}(f_s)$, and $\beta_s(\tau_0)$ does not enumerate any initial segment of τ_0.

The reader unfamiliar with priority tree arguments can find a description of this method in [42]. Our priority tree will be a subtree of the binary tree. We assign the requirement R_e to all nodes on level $2e$, and the requirement P_e to nodes on level $2e + 1$. Each node assigned to an R-requirement shall have two outcomes, $\infty <_L fin$, while each node assigned to a P-requirement has only one outcome 0. Each node η assigned to a P-requirement shall have a parameter x_η, where it will try and meet its requirement with $f([x_\eta])$. A node η assigned to an R-requirement shall define a c.e. set W_η and attempt to satisfy its requirement by making $\Phi_e\left(\mathtt{Name}(f)\right) \subseteq \mathtt{Enum}(W_\eta)$.

Construction. At stage s, we define δ_s, the approximation to the true path of the construction. As usual, we say that η is visited at stage s if $\delta_s \supset \eta$. To initialize a node means to reset all parameters associated with the node. At stage s, assume that $\eta \subseteq \delta_s$ has been defined. If $|\eta| < s$ we describe the actions taken by η.

Suppose η is assigned a P-requirement. If $x_\eta \uparrow$ we pick a fresh new value for x_η. Otherwise, if $x_\eta \downarrow$ but η is not yet satisfied, and φ_η has enumerated some element $\langle \tau, 2x_\eta\rangle$ for some $\tau \supseteq x_\eta$ during the same stage as or before the previous visit to η, we change $f_s(x_\eta)$ from 0 to $\tau{}^\frown j$ for a fresh number j. Declare η as now being *satisfied*, and initialize all nodes extending η. Otherwise do nothing for η. Let $\delta_s(|\eta|) = 0$.

Suppose instead that η is assigned an R-requirement. Let $t < s$ be the previous stage where we visited $\eta^\frown\infty$ (obviously $t = 0$ if no previous stage exists). Let $\tau_0, \cdots, \tau_n$ be a list of all nodes τ such that $\langle \tau, 2x_\nu \rangle$ has been enumerated by φ_ν during or before stage t, for some $\tau \supseteq x_\nu$ and $\nu \supseteq \eta^\frown\infty$ where ν is not yet satisfied. (Obviously, for each ν, if φ_ν has enumerated more than one such τ, consider only the first τ to be enumerated by φ_ν). If $\Phi_\eta(\beta_s(\tau_i))$ has enumerated all (the finitely many) elements of $W_{\eta,s}$ for each $i \leq n$, we take $\delta_s(|\eta|) = \infty$, otherwise take $\delta_s(|\eta|) = fin$.

Finally, if $|\eta| = s$, we take $\delta_s = \eta$, and initialize all nodes to the right of δ_s. If no P-requirement has been declared satisfied at this stage, we do the following for each $\eta \subset \delta_s$ such that $\eta^\frown\infty \subseteq \delta_s$ and η is assigned an R-requirement. If there is a smallest new element $k \notin W_{\eta,s}$ such that $k \in \Phi_\eta(\alpha_s)$, we add k to $W_{\eta,s+1}$. If some P-requirement has managed to become satisfied at this stage, we do not increase W for any R-requirement and go to the next stage of the construction.

3.6.1. *Verification.* We now verify that the construction satisfies all requirements. Let $\delta = \liminf_s \delta_s$ be the true path of the construction. First of all, it is obvious that each node on the true path is initialized only finitely often.

Now for each n, we argue that there are infinitely many s such that $\delta \upharpoonright n$ is visited at s and we increase W at (the end of) stage s. Suppose not. We eventually never visit left of $\delta \upharpoonright n$, and all P-nodes $\eta \subset \delta \upharpoonright n$ will eventually stop acting. This means that eventually at every visit to $\delta \upharpoonright n$, some node $\eta \supset \delta \upharpoonright n$ must be made satisfied during that visit. But every time we satisfy some P-node η, we will also at the same time initialize every node strictly extending or to the right of η. That means that eventually, at some visit to $\delta \upharpoonright n$, every P-node along δ_s extending $\delta \upharpoonright n$ must have no follower assigned during that stage (or is already satisfied). At such a stage, no P-requirement can be made satisfied, a contradiction.

We now argue that all P-requirements are satisfied along the true path. Fix a P-node $\eta \subset \delta$, and suppose that φ_η is total and in $\mathtt{Name}(f)$. Let x_η be the final follower picked by η. Since x_η is picked fresh, and if η is never satisfied, then $f_s(x_\eta) = 0$ for all s, which means that $f([x_\eta]) = \{2x_\eta\}$. This means that η will be declared satisfied at the second visit to η. We would also switch $f(x_\eta) = \tau^\frown j$ for some τ and j such that $\langle \tau, 2x_\eta \rangle \in rng(\varphi_\eta)$, which means that φ_η cannot be in $\mathtt{Name}(f)$ after all.

We now argue that all R-requirements are satisfied along the true path. Fix an R-node $\eta \subset \delta$ and assume that there is some A such that $\Phi_e(\mathtt{Name}(f)) \subseteq \mathtt{Enum}(A)$. We proceed in several steps:

(i) First we argue that for every s and every $k \in W_{\eta,s}$, $k \in \Phi_\eta(\alpha_s)$. We fix k and proceed by an induction on s. When k is initially enumerated in $W_{\eta,s+1}$ at the end of some stage s, we of course have $k \in \Phi_\eta(\alpha_s)$. Subsequently f is only modified by a P-requirement extending η or to the right of η. Nodes to the right of $\eta^\frown\infty$ pick their followers fresh after this, so their actions cannot cause k to leave $\Phi_\eta(\alpha_s)$. If ν is a P-node extending $\eta^\frown\infty$ then ν only modifies $f(x_\nu)$ whenever $\eta^\frown\infty$ is visited; at such a stage (and owing to the fact that ν only modifies $f(x_\nu)$ the second time it is visited after it first discovers that φ_ν has enumerated some $\langle \tau, 2x_\nu \rangle$) we must have $k \in \Phi_\eta(\beta_s(\tau))$. At this stage ν is allowed to modify $f(x_\nu) = \tau^\frown j$ for a fresh j. As j is fresh, it is much larger than any axiom involved in $k \in \Phi_\eta(\beta_s(\tau))$, and since $\beta_s(\tau)$ does not enumerate any initial segment of τ, this means that after this modification to f we will still have $k \in \Phi_\eta(\alpha_s)$.

(ii) Next we argue that $\eta^\frown\infty \subset \delta$; suppose not. Then we visit $\eta^\frown\infty$ only finitely often and the final value of t mentioned in the strategy for R exists. The corresponding list of $\tau_0, \cdots, \tau_n$ would also be stable (associated with the nodes $\nu_0, \cdots, \nu_n \supseteq \eta^\frown\infty$). But as we never attend to any node extending $\eta^\frown\infty$ after stage t, this means that $f(x_{\nu_i}) = 0$ for every $i \leq n$. Therefore, $\beta_s(\tau_i) \in \mathtt{Name}(f_s)$ for every s and every $i \leq n$. For each of the finitely many $k \in W_\eta$, and by item (i) above, $k \in \Phi_\eta(\alpha_s)$ for every s. Note that this fact in itself is of course not enough to guarantee that $k \in \Phi_\eta(\lim_s \alpha_s)$, but our usual convention on the choice of fresh followers for a P-requirement will ensure that this holds at the end. Since $\lim_s \alpha_s \in \mathtt{Name}(f)$, this means that $k \in A$. Since $\lim_s \beta_s(\tau_i) \in \mathtt{Name}(f)$ for every $i \leq n$, that means that k must be enumerated by $\Phi_\eta(\lim_s \beta_s(\tau_i))$ for every $i \leq n$. This must be witnessed at a finite stage, a contradiction. (If $t = 0$, then $W_\eta = \emptyset$ and it is trivial).

(iii) Finally we argue that $W_\eta = A$. By (ii), $\eta^\frown\infty$ is along the true path. Suppose $k \in A$. Then $k \in \Phi_\eta(\alpha_s)$ for all large enough s. But we have already verified that there are infinitely many stages where $\eta^\frown\infty$ is visited and W_η is increased. Hence $k \in W_\eta$. Now suppose that $k \in W_\eta$. Then by item (i), $k \in \Phi_\eta(\alpha_s)$ for every s. This means that $k \in A$.

This ends the proof of Theorem 3.13. □

Remark. The above proof only uses the property mentioned in Observation 2.28. Hence, our proof also shows that there exists a $C(\mathbb{Q})$-degree which is quasiminimal with respect to all e-degrees.

Note that Theorem 3.13 implies an $\mathcal{O}(\omega^\omega)$-degree which is quasiminimal w.r.t. e-degrees by the following observation:

Observation 3.14. $C(\omega^\omega, \omega) \subseteq C(\omega^\omega, \omega_{\rm cof}) \subseteq C(\omega^\omega, \mathbb{S})$.

Proof. For the first inclusion, one can see that $C(\omega^\omega, \omega) \subseteq C(\omega^\omega, 2^\omega)$ by Observation 1.3. Note that $C(X \times Y, Z) \simeq C(X, C(Y, Z))$, where $\simeq$ indicates that these spaces are computably homeomorphic. Therefore, $C(\omega^\omega, 2^\omega) \simeq C(\omega^\omega \times \omega, 2) \simeq C(\omega^\omega, 2) \simeq C(\omega^\omega, 2_{\rm cof}) \subseteq C(\omega^\omega, \omega_{\rm cof})$. Consequently $C(\omega^\omega, \omega)$ computably embeds into $C(\omega^\omega, \omega_{\rm cof})$. The second inclusion follows from Corollary 1.4. □

The above proof actually shows that if X is a higher-order Kleene-Kreisel space, then $C(X, \omega) \subseteq C(X, \omega_{\rm cof}) \subseteq C(X, \mathbb{S})$ holds. Note that $C(X, \omega)$ is Hausdorff, $C(X, \omega_{\rm cof})$ is T_1, $\omega^\omega_{\rm co} \subseteq C(X, \omega_{\rm cof})$, and $\mathbb{R}_< \subseteq C(X, \mathbb{S})$. Therefore, the inclusions are proper (even in the degree-theoretic sense) since there are a $C(\omega^\omega, \omega)$-quasiminimal $\omega^\omega_{\rm co}$-degree, and a T_1-quasiminimal $\mathbb{R}_<$-degree, by [22].

Now, we wish to investigate the "second countable fragment" of each properly third order space. That is, given $C(X, Y)$, we wish to investigate the class of all sets A such that $\mathtt{Enum}(A) = \mathtt{Name}(F)$ for some $F \in C(X, Y)$; in this case we say that the space $C(X, Y)$ *realizes* the e-degree of A. For instance, it is easy to see that $\mathcal{O}(\omega^\omega)$ realizes all e-degrees. On the other hand, there is an $\omega^\omega_{\rm co}$-degree which is quasiminimal with respect to $\omega\langle 2\rangle$ as mentioned in the previous paragraph. Hence, it is natural to ask which e-degrees are realized by $\omega\langle 2\rangle$.

Here, we give a partial result saying that there is a cone of $\omega\langle k\rangle$ which only realizes total e-degrees.

Proposition 3.15. *There is $r \in 2^\omega$ such that, for any k, if $f \in \omega\langle k\rangle$ computes r and has an e-degree, then f has a total degree.*

Proof. A topological space is *quasi-zero-dimensional* if it is the sequential coreflection of a zero-dimensional space. Schröder [40] showed that the Kleene-Kreisel space $\omega\langle k\rangle$ is quasi-zero-dimensional, and moreover, every qcb subspace of a quasi-zero-dimensional qcb space is also quasi-zero-dimensional. Hence, every second-countable subspace of $\omega\langle k\rangle$ is zero-dimensional as every second-countable space is sequential. Assume that $f \in \omega\langle k\rangle$ has an e-degree, say $f \equiv_T A \subseteq \omega$ via Φ and Ψ. Since $S_{\Phi,\Psi} = \{g \in \omega\langle k\rangle : \Phi \circ \Psi(g) = g\}$ is homeomorphic to a subspace of the second-countable space $\mathcal{O}(\omega)$, it is zero-dimensional by the above argument. Moreover, as $S_{\Phi,\Psi}$ is a subspace of $\omega\langle k\rangle$, it is Hausdorff, and by zero-dimensionality, indeed, it is metrizable. Therefore, $S_{\Phi,\Psi}$ can be embedded into ω^ω. Hence, every $g \in S_{\Phi,\Psi}$ (in particular, f) has a total degree relative to some oracle $r_{\Phi,\Psi}$ depending on Φ and Ψ. Let $r \in 2^\omega$ be the supremum (w.r.t. Turing reducibility) of all such $r_{\Phi,\Psi}$ for any pair Φ, Ψ. This implies that, if $f \in \omega\langle k\rangle$ has an e-degree, then f is total relative to r, that is, $f \oplus r \equiv_T x$ for some $x \in 2^\omega$. □

An open question is to calculate the exact complexity of such an oracle r.

3.7. **The third order space with cofinite topology: $C(C(\omega_{\rm cof}), \omega_{\rm cof})$.** We here generalize the construction of Kleene-Kreisel spaces. We define

$$X\langle 0\rangle = X, \qquad\qquad X\langle n+1\rangle = C(X\langle n\rangle, X)$$

for any n. We consider the type hierarchy over ground type $X = \omega_{\rm cof}$. As we have seen in Proposition 3.4, $\omega_{\rm cof}\langle 1\rangle$ is a second countable space. Thus, we next consider $\omega_{\rm cof}\langle 2\rangle$. Note that N is a local network at $F \in \omega_{\rm cof}\langle 2\rangle$ iff

$$(\forall g)(\forall n)\ [F(g) \neq n \ \leftrightarrow\ (\exists D = \langle A_i, a_i\rangle_{i<k})\ ((\forall i < k)\ g^{-1}\{a_i\} \subseteq A_i \text{ and } \langle D, n\rangle \in N)]$$

The equivalence inside the above square bracket can be rewritten as follows:

$$F(g) = n \ \leftrightarrow\ (\forall D = \langle A_i, a_i\rangle_{i<k})\ [\langle D, n\rangle \in N \ \to\ (\exists i < k)\ g^{-1}\{a_i\} \not\subseteq A_i].$$

Definition 3.16. We define $\mathcal{G}(A, a) = \{g \in \omega_{\rm cof}\langle 1\rangle : g^{-1}\{a\} \not\subseteq A\}$, and $\mathcal{G}(D) = \bigcup_{(A,a)\in D} \mathcal{G}(A, a)$.

Then, a name for $F \in \omega_{\rm cof}\langle 2\rangle$ can be considered as an enumeration of basic information $\langle D, n\rangle$ specifying $F^{-1}\{n\} \subseteq \mathcal{G}(D)$.

Now, the difficulty is that a basic $\omega_{\rm cof}\langle 1\rangle$-closed set $\mathcal{G}(D)$ is not ω^ω-clopen. Indeed, $\mathcal{G}(D)$ is quite large, so that $\mathcal{G}(D) \cap \mathcal{G}(E) \neq \emptyset$ for any D and E. This means that we cannot ensure that $F^{-1}\{n\}$ is of the form $\mathcal{G}(D)$ for all but one n.

We now need to consider what kind of sets are closed in $\omega_{\rm cof}\langle 1\rangle$. For example, $[\sigma] = \{g \in \omega_{\rm cof}\langle 1\rangle : \sigma \prec g\}$ is closed, since $[\sigma] = \bigcap_{i<|\sigma|} \bigcap_e \mathcal{G}(D_e \setminus \{i\}, \sigma(i))$. Note that we still have $\mathcal{G}(D) \cap [\sigma] \neq \emptyset$. However, the point is that $[\sigma] \subseteq \mathcal{G}(D)$ for some σ.

Observation 3.17. *$\omega_{\rm cof}\langle 1\rangle$ computably embeds into $\omega_{\rm cof}\langle 2\rangle$.*

Proof. Given $g\colon \omega_{\rm cof}\langle 1\rangle$, define $G(n * x) = g(n)$. Then, $g^{-1}\{n\} \subseteq D$ if and only if $G^{-1}\{n\} \subseteq \bigcup_{k\in D} [k]$. □

Kihara-Ng-Pauly [22] have shown that there is an $(\omega^\omega_{\rm co})$-degree which is $\omega\langle k\rangle$-quasiminimal for any k. Hence, by Proposition 3.5, there is an $\omega_{\rm cof}\langle 1\rangle$-degree which is $\omega\langle k\rangle$-quasiminimal. Therefore, by Observation 3.17, we also have an $\omega_{\rm cof}\langle 2\rangle$-degree which is $\omega\langle k\rangle$-quasiminimal.

Theorem 3.18. *There is an $\omega_{\rm cof}\langle 2\rangle$-degree which is not an e-degree.*

Proof. We will define $H \in \omega_{\rm cof}\langle 2\rangle$, by declaring a sequence $T = (T_k)$ of cylinders such that $H^{-1}\{k\} = T_k$.

Let $\langle \Phi_s, \Psi_s\rangle_{s\in\omega}$ be an enumeration of all pairs of partial computable functions. At stage s, we attempt to ensure that $\Phi_s\Psi_s(H) \neq H$ whenever $\Psi_s(H) \in \mathbb{S}^\omega$. Assume that we have constructed a co-infinite set Λ_s, and a sequence $(T_t)_{t\in\Lambda_s}$ of cylinders, where we also assume that $(T_t)_{t\in\Lambda_s}$ covers $\{\langle r\rangle\}_{r<s}$, but have only finitely many cylinders in the outside of $\{\langle r\rangle\}_{r<s}$. Let u be the least element not in Λ_s, and choose σ_s which is not covered by $(T_t)_{t\in\Lambda_s}$.

Case 1. Assume that for any $G \in \omega_{\rm cof}\langle 2\rangle$ preserving the previous declaration, if $G^{-1}\{u\} \subseteq [\sigma_s]$, and there is $D \subseteq \Psi_s(G)$ such that $\langle E, u\rangle \in \Phi_s(D)$, then $[\sigma_s] \subseteq \mathcal{G}(E)$.

In this case, put $T_u = [\sigma_s * 0]$. This ensures that $H^{-1}\{u\} \subsetneq [\sigma_s]$, which eventually guarantees $H \neq \Phi_s\Psi_s(H)$.

For the sake of totality of H, if $\langle s\rangle$ is not covered by $(T_t)_{t\in\Lambda_s}$, let $(\tau_i)_{i\in\omega}$ be a pairwise incomparable sequence of strings not covered by $(T_t)_{t\in\Lambda_s}$ and T_u, but the union of T_u, $\{T_t\}_{t\in\Lambda_s}$, and $(\tau_i)_{i\in\omega}$ covers $\langle s\rangle$. Choose a sparse infinite sequence $\{v_i\}_{i\in\omega} \subseteq \omega \setminus \Lambda_s$, and define $T_{v_n} = [\tau_n]$. Put $\Lambda_{s+1} = \Lambda_s \cup \{u\} \cup \{v_i\}_{i\in\omega}$.

Case 2. Otherwise, there are $G \in \omega_{\rm cof}\langle 2\rangle$ preserving $(T_t)_{t\in\Lambda_s}$, $G^{-1}\{u\} \subseteq [\sigma_s]$, and some $D \subseteq \Psi_s(G)$ such that $\langle E, u\rangle \in \Phi_s(D)$ and $[\sigma_s] \not\subseteq \mathcal{G}(E)$.

In this case, choose such a G. Let $k \neq u$ be a large number which is not mentioned in E. A name of G is a sequence $\langle C_j, n_j\rangle_{j\in\omega}$. Let $(t(i))_{i\in\omega}$ be an increasing enumeration of all j's such that $n_j = u$, and define $C^*_{j,e} = C_j \cup \{\langle D_e, k\rangle\}$ if $n_j = u$. Then, we have

$$[\sigma_s] \cap \bigcap_{\langle j,e\rangle<i} \mathcal{G}(C^*_{t(j),e}) \supseteq [\sigma_s] \cap \bigcap_{\langle j,e\rangle<i} \mathcal{G}(D_e, k) \not\subseteq \mathcal{G}(E)$$

for any $i \in \omega$. Note that the last non-inclusion follows from the assumption that $[\sigma_s] \not\subseteq \mathcal{G}(E)$ and k is not mentioned in E.

We now claim that $G^{-1}\{u\} = [\sigma_s] \cap \bigcap_{j,e\in\omega} \mathcal{G}(C^*_{t(j),e})$. The inclusion $\subseteq$ is clear since we have $G^{-1}\{u\} = \bigcap_j \mathcal{G}(C_{t(j)}) \subseteq [\sigma_s]$. For the reverse inclusion $\supseteq$, suppose not. Then, there is $g \in \omega_{\rm cof}\langle 1\rangle$ such that $g \in [\sigma_s]$ and $g \in \mathcal{G}(C^*_{t(j),e})$ for any $j, e \in \omega$, but $g \notin \mathcal{G}(C_{t(j)})$ for some j. Choose such an j. By our choice of g, we have $g \in \mathcal{G}(C_{t(j)}) \cup \mathcal{G}(D_e, k)$, and therefore $g \in \mathcal{G}(D_e, k)$, for any $e \in \omega$. Recall that every $g \in \omega_{\rm cof}\langle 1\rangle$ is either constant or finite-to-one. However, $g \in \mathcal{G}(D_e, k)$ implies that $g^{-1}\{k\} \not\subseteq D_e$, for any $e \in \omega$, and therefore, g cannot be finite-to-one. Consequently, $g(i) = k$ for any $i \in \omega$. However, as $g \in [\sigma_s]$, we have $g(i) = \sigma_s(i)$ for any $i < |\sigma_s|$, but it is impossible since we also have $\sigma_s(i) \neq k$. This verifies the claim.

This procedure gives us an infinite decreasing sequence $(\mathcal{G}_n)$ of finitely generated sets such that

$$\mathcal{G}(A, a) = \mathcal{G}_0 \supseteq \mathcal{G}_1 \supseteq \mathcal{G}_2 \supseteq \cdots \to G^{-1}\{u\},$$

and $\mathcal{G}_n \not\subseteq \mathcal{G}(E)$ for any $n \in \omega$, where the arrow indicates that $\bigcap_n \mathcal{G}_n = G^{-1}\{u\}$. Consider a name of G given by this slowly converging sequence $p = (\mathcal{G}_n)$. Then a finite initial sequent of p already witnesses $D \subseteq \Psi_s(G)$. So, except for information on Λ_s, finitely many information $(T_t)_{t\in F}$ is used to ensure $D \subseteq \Psi_s(h)$, where $[T_u] \not\subseteq [E]$. By extending a partial name, we can assume that T_t is a cylinder for each $t \in F \cup \{u\}$, and that the collection $(T_t)_{t\in\Lambda_s\cup F}$ is pairwise disjoint.

Choose $(\tau_i)_{i\in\omega}$ and $\{v_i\}_{i\in\omega} \subseteq \omega \setminus \Lambda_s \cup F$ in the same manner as Case 1. Define $T_{v_n} = [\tau_n]$, and put $\Lambda_{s+1} = \Lambda_s \cup F \cup \{v_i\}_{i\in\omega}$. □

3.8. **Higher order spaces.** As $\omega\langle k\rangle$ has a countable network, it is clearly separable. For effectivity, it is known that there is a primitive recursive sequence $(q^k_n)_{n\in\omega}$ which is dense in $\omega\langle k\rangle$ (see Normann [33, Chapter 5]). A *trace* of $\psi \in \omega\langle k+1\rangle$ is a function $h_\psi \colon \omega \to \omega$ defined by $h_\psi(n) = \psi(q^k_n)$. By continuity of $\psi \in \omega\langle k+1\rangle$, one can show that the principal associate (which is a special kind of a name) of ψ is computable in the jump of the trace h_ψ (see Normann [33, Theorem 5.15]). As both the trace and the principal associate are elements of ω^ω, it is easy to conclude the following:

Proposition 3.19. *For any k, there is a continuous degree which is not an $\omega\langle k\rangle$-degree. In particular, for any $k > 0$, the collection of $\mathbb{R}\langle k\rangle$-degrees are strictly larger than the $\omega\langle k\rangle$-degrees.*

Proof. Let $\psi \in \omega\langle k\rangle$ be given. As mentioned above, by Normann [33, Theorem 5.15], there are $h_\psi, p_\psi \in \omega^\omega$ such that $h_\psi \leq_T \psi \leq_T p_\psi \leq_T h'_\psi$. In particular, the lower cone $\{x \in \omega^\omega : x \leq_T \psi\}$ is not closed under the Turing jump. As mentioned in the proof of Proposition 3.9, Miller [31] showed that every countable Scott ideal is of the form $\{x \in \omega^\omega : x \leq_T f\}$ for some $f \in [0,1]^\omega$. So take any countable jump ideal (i.e., any Turing ideal closed under the Turing jump), which is in particular a Scott ideal. This shows that there is a continuous degree which is not an $\omega\langle k\rangle$-degree.

For the second assertion, it is easy to check that $[0,1]^\omega \subseteq \mathbb{R}\langle k\rangle$ for any $k > 0$. □

Recall that a topological space X is Γ-representable (cf. Schröder-Selivanov [41]) if X has an admissible representation δ such that $\mathrm{Eq}(\delta) = \{(p,q) : p,q \in \mathrm{dom}(\delta) \text{ and } \delta(p) = \delta(q)\}$ is in $\Gamma(\omega^\omega)$.

Let $\mathsf{CB}_0(\utilde{\mathbf{\Delta}}^1_1)$ be the collection of all Borel-representable second-countable T_0-spaces, that is, the spaces which are homeomorphic to a Borel subset of the universal second-countable T_0-space $\mathbb{S}^\omega$. Then, we define $\mathsf{CB}_0\langle 0\rangle = \mathsf{CB}_0(\utilde{\mathbf{\Delta}}^1_1)$, and let $\mathsf{CB}_0\langle k+1\rangle$ be the collection of the spaces of the form $C(X,Y)$ for some $X, Y \in \mathsf{CB}_0\langle k\rangle$. For instance, if X is a Borel-representable second-countable T_0-space, then $X\langle k\rangle \in \mathsf{CB}_0\langle k\rangle$. In particular, $\omega\langle k+1\rangle, \omega_{\mathrm{cof}}\langle k+1\rangle \in \mathsf{CB}_0\langle k\rangle$.

Proposition 3.20 (Schröder-Selivanov [41, Proposition 4.3])**.** *For any $1 \leq \alpha < \omega_1$, if X is $\utilde{\mathbf{\Sigma}}^1_\alpha$-representable, and if Y is $\utilde{\mathbf{\Pi}}^1_\alpha$-representable, then $C(X,Y)$ is $\utilde{\mathbf{\Pi}}^1_\alpha$-representable.*

In particular, for any $k > 0$, every $\mathsf{CB}_0\langle k\rangle$-space is $\utilde{\mathbf{\Pi}}^1_k$-representable. □

Hinman [13] showed that there is an $\omega\langle 2\rangle$-degree which is not an $\omega\langle 1\rangle$-degree. Then Dvornickov (cf. Normann [33, Corollary 7.2]) extended his result by showing that for any k, there is an $\omega\langle k+1\rangle$-degree which is not an $\omega\langle k\rangle$-degree.

Dvornickov's proof of [33, Theorem 7.1] only uses the fact that the set of all associates (i.e., names) of $\psi \in \omega\langle k-1\rangle$ is $\utilde{\mathbf{\Pi}}^1_{k-2}$, which follows from the fact that $\omega\langle k-1\rangle$ is $\utilde{\mathbf{\Pi}}^1_{k-2}$-representable, and then applies a technical lemma [33, Lemma 5.31]. Hence, the proof is applicable for any $\utilde{\mathbf{\Pi}}^1_{k-2}$-representable space (indeed, any $\utilde{\mathbf{\Sigma}}^1_{k-1}$-representable space), so it actually shows the following:

Theorem 3.21. *There is an $\omega\langle k+2\rangle$-degree which is not an X-degree for any $\utilde{\mathbf{\Sigma}}^1_{k+1}$-representable space X. In particular, there is an $\omega\langle k+2\rangle$-degree which is not a $\mathsf{CB}_0\langle k\rangle$-degree.* □

As a corollary, we obtain an $\omega\langle k+2\rangle$-degree which is not an $\omega\langle k+1\rangle$-degree, not an $\mathcal{O}(\omega\langle k\rangle)$-degree, and so on.

3.9. **Bounded linear operators.** The function space construction plays a key role in functional analysis. Thus, it is natural to ask if there is an example in functional analysis whose degree structure is nontrivial. For instance, ℓ^∞ is an important example of a non-separable Banach space. Similarly, $B(H)$, the space of bounded linear operators on the infinite dimensional separable Hilbert space $H = \ell^2$, is also non-separable w.r.t. the norm topology induced by the operator norm, since ℓ^∞ isometrically embeds into $B(H)$. Even though ℓ^∞ and $B(H)$ are non-separable (w.r.t. the norm topology), there are known ways of handling with ℓ^∞ and $B(H)$ in computability theory [5, 32].

First, ℓ^∞ is known to be isometrically isomorphic to the dual $(\ell^1)'$ of ℓ^1 by Landau's theorem. As a dual, one can introduce the weak* topology on $\ell^\infty = (\ell^1)'$, which is separable, but not second-countable. Similarly, the most commonly-used topology on the bounded linear operators, $B(H)$, is not the norm topology, but the strong operator topology. The strong operator topology on $B(H)$ is separable, but not second-countable.

Brattka-Schröder [5] showed that the co-restriction of the function space representation to the dual of a represented separable Banach space is admissible w.r.t. weak* topology. More generally, Neumann-Pape-Streicher [32] showed that, for represented separable Banach spaces E and F, the co-restriction of the function space representation to the bounded linear operators, $B(E,F)$, is admissible w.r.t. the strong operator topology.

First we see that the dual of a computable normed space does not add a new degree. Fix $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Recall that the dual X' of a computable normed space X is the space of bounded linear functionals $f\colon X \to \mathbb{F}$, whose representation is inherited from the represented function space $C(X, \mathbb{F})$.

Proposition 3.22. *If X is a computable normed space, then every element of the dual X' has a continuous degree. In particular, the ℓ^∞-degrees are exactly the continuous degrees.*

Proof. Fix $f \in X'$. Since f is bounded, there is b such that $||f|| = \sup_{||x||\le 1} |f(x)| \le b$. By the computable Banach-Alaoglu theorem (see [3]), the bounded ball $B_{X'} = \{g \in X' : ||g|| \le b\}$ is computably embedded into a computably compact metric space as a Π^0_1 subset. As $f \in B_{X'}$, this clearly implies that f has a continuous degree. □

One can also show the similar result for $B(H)$, whose representation is also inherited from the represented function space $C(H)$.

Proposition 3.23. *The $B(H)$-degrees are exactly the continuous degrees.*

Proof. Let $\{x_n\}$ be a computable dense sequence in H. Define a metric on the unit ball on $B(H)$ by $d(S,T) = \sum_n 2^{-n}||(S-T)x_n||_2$. This metric is compatible with the strong operator topology on $B(H)$ on the unit ball [32]. The proof is effective. □

4. Π^0_1 classes and Π^0_1 singletons

4.1. Π^0_1 singletons in $C(\omega_{\mathrm{cof}})$. In this section, we investigate Π^0_1 *singletons* in function spaces. Here, a closed set in a represented space X is Π^0_1 if its complement is a computable point in $\mathcal{O}(X)$. We often identify a Π^0_1 singleton with its unique element; that is, it is $x \in X$ such that $X \setminus \{x\}$ is a computable point in $\mathcal{O}(X)$. For $X \in \{2^\omega, \omega^\omega\}$, this notion has been studied in depth in classical computability theory [35, Definition XII.2.13].

From a computability-theoretic perspective, giving a name of a Π^0_1 singleton $\{x\}$ (in a second-countable space) is determining a point x by enumerating its negative information, which may play an important role in understanding computability with negative data. In addition, this notion has a topologically meaning: As pointed out by Kihara-Pauly [24], the notion of a Π^0_1 singleton can be understood as *de Groot dual* in general topology; see also Section 4.2 (For example, ω_{cof} is the de Groot dual of ω).

We now discuss basis theorems for Π^0_1 singletons in function spaces. A represented space X is Γ-*named* if the domain of its representation is in Γ. First, one can easily generalize the known fact [35, Proposition XII.2.16] that every Π^0_1 singleton in ω^ω is hyperarithmetic as follows:

Proposition 4.1. *Let X be a represented cb_0 space which is Σ^1_1-named. Then, every Π^0_1 singleton in X is hyperarithmetic.*

Proof. Let $(B_n)_{n\in\omega}$ be a computable basis of X such that the domain of the induced representation $\delta\colon\subseteq \omega^\omega \to X$ is Σ^1_1, that is, the set of all enumerations p of $\mathtt{Nbase}(x)$ for some $x \in X$ is Σ^1_1. Let $\{x\}$ be a Π^0_1 singleton in X, and then there is computable $\varphi\colon\subseteq \omega^\omega \to \mathbf{1}$ such that $z \in \{x\}$ iff for any (some) name p of z, $\varphi(p)\uparrow$. Thus, the unique element x in $\{x\}$ satisfies the following property:

$$\begin{aligned} x \in B_n &\iff (\forall p)\ [\text{if } p \in \mathrm{dom}(\delta) \text{ and } \varphi(p)\uparrow,\ \text{then } (\exists i)\ p(i) = n] \\ &\iff (\exists p)\ [p \in \mathrm{dom}(\delta),\ \varphi(p)\uparrow \text{ and } (\exists i)\ p(i) = n]. \end{aligned}$$

Clearly, the statement in the first line is Π^1_1, and the second line is Σ^1_1. Therefore, $\mathtt{Nbase}(x)$ is Δ^1_1. Consequently, x has a hyperarithmetical name. □

Corollary 4.2. *Every Π^0_1 singleton in $C(\omega_{\mathrm{cof}})$ is hyperarithmetic.* □

It is known that the degrees of Π^0_1 singletons in the space ω^ω (i.e. $C(\omega)$) is cofinal in the hyperarithmetical hierarchy [35, Proposition XII.2.19]. If we consider the subspace $C_b(\omega)$ of ω^ω consisting of computably bounded functions, it is clear that every Π^0_1 singleton in $C_b(\omega)$ is computable [35, Exercise XII.2.15 (c)]. On the contrary, we show that the degrees of Π^0_1 singletons in the space $C_b(\omega_{\mathrm{cof}})$ of computably bounded functions is cofinal in the hyperarithmetical hierarchy.

Theorem 4.3. *For any computable ordinal α, there is a Π^0_1 singleton $\{g\}$ in $C_b(\omega_{\mathrm{cof}})$ such that $\emptyset^{(\alpha)} \leq_T g$ holds.*

To show this, we need a few lemmas. Let $\mathbb{S}^\neg$ be the negation of the Sierpiński space, that is, $\emptyset, \{0\}, \{0,1\}$ are open. Then define $\utilde{\mathbf{\Pi}}^0_1(X) = C(X, \mathbb{S}^\neg)$. Note that we view $\utilde{\mathbf{\Pi}}^0_1(X)$ as the hyperspace of closed subsets of X by identifying a set with its characteristic function.

Given a set $P \subseteq X$, and a function $Q\colon X \to \utilde{\mathbf{\Pi}}^0_1(Y)$, define $Q \star P = \{(x,y) : x \in P \text{ and } y \in Q(x)\}$.

Lemma 4.4. *If P is Π^0_1 in X, and $Q\colon X \to \underset{\sim}{\mathbf{\Pi}}^0_1(Y)$ is computable, then $Q \star P$ is Π^0_1 in $X \times Y$.*

Proof. Let φ be a computable realizer of P, and ψ be a computable realizer of Q. Given (p,q), wait for seeing that either $\varphi(p)$ is rejected, or $\psi(p)(q)$ is rejected. If this happens, we reject (p,q). This procedure gives a computable realizer of $Q \star P$. $\square$

Recall that every point $g \in C(\omega_{\rm cof})$ is finite-to-one. Note that even if g_n is finite-to-one for any $n \in \omega$, the join $\bigoplus_n g_n(\langle u,v\rangle) = g_u(v)$ is not necessarily finite-to-one. Instead, we consider $\sqcup_n g_n$ defined by $(\sqcup_n g_n)(\langle u,v\rangle) = \langle u, g_u(v)\rangle$. It is clear that $\sqcup_n g_n$ is finite-to-one (computably bounded, resp.) whenever g_n is finite-to-one (computably bounded, resp.) for any $n \in \omega$.

Lemma 4.5. *If $(P_n)_{n\in\omega}$ is a computable sequence of Π^0_1 singletons in $C(\omega_{\rm cof})$, then $\{\sqcup_n g_n : (\forall n)\ g_n \in P_n\}$ is also Π^0_1 in $C(\omega_{\rm cof})$.*

Proof. Assume that P_n is of the form $\bigcap_{D\in F_n} \mathcal{G}(D)$ for a c.e. set F_n uniformly in $n \in \omega$. Our basic idea is, for $(A,a) \in D \in F_n$, instead of considering $\mathcal{G}(A,a) = \{g : g^{-1}\{a\} \not\subseteq A\}$, to consider the following for any s:

$$g^{-1}\{\langle n,a\rangle\} \not\subseteq \{\langle n,i\rangle : i \in A\} \cup \{\langle m,i\rangle : m \neq n \text{ and } \langle m,i\rangle < s\}.$$

Note that requiring g to satisfy the above condition for any s eventually forces $g^{-1}\{\langle n,a\rangle\} \not\subseteq \{\langle n,i\rangle : i \in A\}$ by finite-to-oneness of g. Below we give a precise description of this idea.

Let $(D^n_k)_{k\in\omega}$ be a computable enumeration of F_n. Recall that D^n_k is a finite sequence $(A^n_{k,j}, a^n_{k,j})_{j<\ell}$. Then, we define $B^{n,s}_{k,j} = \{\langle n,i\rangle : i \in A^n_{k,j}\} \cup \{\langle m,i\rangle : m \neq n \text{ and } \langle m,i\rangle < s\}$. Then, define $E^{n,s}_k = (B^{n,s}_{k,j}, \langle n, a^n_{k,j}\rangle)_{j<\ell}$.

We will show that $\bigcap_{n,s,k} \mathcal{G}(E^{n,s}_k) = \{\sqcup_n g_n : (\forall n)\ g_n \in P_n\}$. Clearly, $\sqcup_n g_n \in \mathcal{G}(E^{n,s}_k)$ whenever $g_n \in P_n$ for all n. For any n,k, by pigeonhole principle, there is j such that $g \in \mathcal{G}(B^{n,s}_{k,j}, \langle n, a^n_{k,j}\rangle)$ for infinitely many s. That is, $g(\langle m,i\rangle) = \langle n, a^n_{k,j}\rangle$ for some $\langle m,i\rangle \notin B^{n,s}_{k,j}$. Note that

$$\langle m,i\rangle \notin B^{n,s}_{k,j} \iff (m = n \ \to\ i \notin A^n_{k,j}) \text{ and } (m \neq n \ \to\ \langle m,i\rangle \geq s).$$

We claim that $g(\langle n,i\rangle) = \langle n, a^n_{k,j}\rangle$ for some $i \notin A^n_{k,j}$. Otherwise, for infinitely many s, we must have $g(\langle m,i\rangle) = \langle n, a^n_{k,j}\rangle$ for some $\langle m,i\rangle \geq s$. However, this contradicts the fact that g is finite-to-one.

Define $g_n(i)$ as the second coordinate of $g(\langle n,i\rangle)$. Then, for any n,k, there is j such that, by the above claim, $g_n(i) = a^n_{k,j}$ for some $i \notin A^n_{k,j}$. This means that $g_n \in P_n$.

It remains to show that $g = \sqcup_n g_n$. Otherwise, there are n,k such that $g(\langle n,k\rangle) = \langle m,a\rangle$ for some $m \neq n$. Define $h(\ell) = g_n(\ell)$ for any $\ell \neq k$ and $h(k) \neq g_n(k)$. Then h is finite-to-one, and $h \neq g_n$. Note that $h \in \mathcal{G}(A^n_{k,j}, a^n_{k,j})$. This is because $g(\langle n,i\rangle) = \langle n, a^n_{k,j}\rangle$ for some $i \notin A^n_{k,j}$ as above; however we have $m \neq n$ and therefore $i \neq k$, so $h(i) = g_n(i) = a^n_{k,j}$. Hence, we get $h \in P_n$. However, $h \neq g_n \in P_n$, which contradicts our assumption that P_n is a singleton. $\square$

Lemma 4.6. *If P is a Π^0_1 singleton in $C(\omega_{\rm cof}) \times C(\omega_{\rm cof})$, then $\{g \sqcup h : (g,h) \in P\}$ is also Π^0_1 in $C(\omega_{\rm cof})$.*

Proof. A basic open set in $C(\omega_{\rm cof})\times C(\omega_{\rm cof})$ is a set $\mathcal{G}(D,E)$ of the following form:

$$\begin{aligned}(g,h)\in\mathcal{G}(D,E)\iff\quad&[(\exists(A,a)\in D)\ g^{-1}\{a\}\not\subseteq A]\\ \text{or }&[(\exists(B,b)\in E)\ h^{-1}\{b\}\not\subseteq B]\end{aligned}$$

Then, instead of considering $\mathcal{G}(D,E)$, we consider the following for each s:

$$\begin{aligned}&(g\sqcup h)^{-1}\{\langle 0,a\rangle\}\not\subseteq\{\langle 0,i\rangle:i\in A\}\cup\{\langle 1,i\rangle:i<s\}\\ \text{or }&(g\sqcup h)^{-1}\{\langle 1,b\rangle\}\not\subseteq\{\langle 1,i\rangle:i\in B\}\cup\{\langle 0,i\rangle:i<s\}.\end{aligned}$$

Now a straightforward modification of the proof of Lemma 4.5 shows the desired assertion. □

Lemma 4.7. *For any $g\in\omega^\omega$, one can effectively find a $\Pi^0_1(g)$ singleton $\{h\}$ in $C_b(\omega_{\rm cof})$ such that $g'\leq_T(g,h)$.*

Proof. Given g, we describe an effective procedure to construct h by enumerating a name of $\{h\}$ in $C_b(\omega_{\rm cof})$. We will also inductively define a parameter ℓ_e, which is a height used to code $g'(e)$.

At stage s, given $e<s$, assume that ℓ_e has already been defined. The e-th strategy waits for $g'(e)=1$. If not at the current stage s, then we guess $h(\ell_e)=2\ell_e$ by enumerating $(2\ell_e,[0,s]\setminus\{\ell_e\})$, which indicates $h^{-1}\{2\ell_e\}\not\subseteq[0,s]\setminus\{\ell_e\}$. Since every point in $C(\omega_{\rm cof})$ is finite-to-one, if we enumerate such a pair for any s, then this eventually determines $h(\ell_e)=2\ell_e$. If we see $g'(e)=1$ at stage s, change our guess to $h(\ell_e)=2\ell_e+1$ by enumerating $(2\ell_e+1,[0,t]\setminus\{\ell_e\})$, which indicates $h^{-1}\{2\ell_e+1\}\not\subseteq[0,t]\setminus\{\ell_e\}$, at any later stage $t\geq s$. Moreover, determine $h\upharpoonright[\ell_e,s]$, say $h(k)=2k$ for $\ell_e<k<s$, ensure $h(s+i)=2\ell_{e+i}+g'(e+i)[s]$ for any $i<s-e$, and injure all lower priority strategies by redefining ℓ_d as $s+d$ for $e<d<s$.

It is not hard to see that this computable procedure eventually determines a singleton $\{h\}$. It is clear that h is computably bounded; indeed, we always have $h(n)\leq 2n+1$. Therefore, from any $C(\omega_{\rm cof})$-name of h, one can recover its positive information as in the proof of Proposition 3.7. Note that $h(\ell_e)=2\ell_e+1$ if and only if $g'(e)=1$. Moreover, given ℓ_e, if $h(\ell_e)=2\ell_e$ then $\ell_{e+1}=\ell_e+1$; otherwise by seeing the stage witnessing $g'(e)=1$, one can compute ℓ_{e+1}. Hence, we can recover $(\ell_e)_{e\in\omega}$ from (g,h). Consequently, we have $g'\leq_T(g,h)$. □

Proof of Theorem 4.3. By Lemma 4.7, there is a Π^0_1 singleton $\{g\}$ in $C(\omega_{\rm cof})$ such that $\emptyset'\leq_T g$, where we always identify g in $C_b(\omega_{\rm cof})$ with the same function in ω^ω by Proposition 3.7. Indeed, Lemma 4.7 ensures that there is a computable function $Q\colon C(\omega_{\rm cof})\to\underset{\sim}{\mathbf{\Pi}}^0_1(C(\omega_{\rm cof}))$ such that $g'\leq_T\langle g,h\rangle$, where $Q(g)=\{h\}$. Note that $Q\star\{g\}=\{(g,h)\}$, which is Π^0_1 in $C(\omega_{\rm cof})\times C(\omega_{\rm cof})$ by Lemma 4.4. Since g and h are finite-to-one, $g\sqcup h$ is also finite-to-one. By Lemma 4.6, one can see that $\{g\sqcup h\}$ is Π^0_1 in $C(\omega_{\rm cof})$. Clearly, $(g,h)\equiv_T g\sqcup h$ since they are total. Therefore, $\emptyset''\leq_T g\sqcup h$, so we get a Π^0_1 singleton in $C(\omega_{\rm cof})$ which computes $\emptyset''$.

By iterating this procedure, one can easily construct a computable sequence of Π^0_1 singletons $\{h_n\}$ in $C(\omega_{\rm cof})$ such that $\emptyset^{(n)}\leq_T h_n$. Then, by Lemma 4.5, $\{\sqcup_n h_n\}$ is also a Π^0_1 singleton in $C(\omega_{\rm cof})$, and by totality of the degree of $\sqcup_n h_n$, it is clear that $\bigoplus_n h_n\leq_T\sqcup_n h_n$. Therefore, we now obtain a Π^0_1 singleton in $C(\omega_{\rm cof})$ which computes $\emptyset^{(\omega)}$. Now, iterate this construction along computable ordinals. □

4.2. **Hyperspace of open sets in $C(\omega_{\rm cof})$.** Now, it is natural to consider the space of co-singletons $X \setminus \{x\} \in \mathcal{O}(X)$. Then a Π^0_1 singleton can be understood as a computable point in such a space. To be more precise, the *de Groot dual* $X^{\sf d}$ of a T_1-space X is the subspace of $\mathcal{O}(X)$ consisting of all co-singletons; see Kihara-Pauly [24] for the background on this notion.

Example 4.8 ([24])**.** $\omega^{\sf d} \simeq \omega_{\rm cof}$, $(\omega_{\rm cof})^{\sf d} \simeq \omega$, and $(\omega^\omega_{\rm co})^{\sf d} \simeq \omega^\omega$. However, $(\omega^\omega)^{\sf d} \not\simeq \omega^\omega_{\rm co}$; indeed, $(\omega^\omega)^{\sf d}$ is not second-countable.

Kihara-Pauly [24] has shown that there is a point in $(\omega^\omega)^{\sf d}$ (i.e., a co-singleton $\omega^\omega \setminus \{x\} \in \mathcal{O}(\omega^\omega)$) which is quasi-minimal w.r.t. e-degrees by using the fact that there is a Π^0_1 singleton in ω^ω which is far from computable. By Theorem 4.3, we now also know that there is a Π^0_1 singleton in $C_b(\omega_{\rm cof})$ (or $C(\omega_{\rm cof})$) which is far from computable. Thus, one can apply the idea in [24] to show the following:

Theorem 4.9. *There is a point in $(C_b(\omega_{\rm cof}))^{\sf d}$ which is quasi-minimal w.r.t. e-degrees.*

For $z \in \omega^\omega$, we say that $g\colon \omega \to \omega$ is *z-bounded-to-one* if there is a z-computable function $b\colon \omega \to \omega$ such that $g^{-1}\{n\} \subseteq [0, b(n)]$ for any $n \in \omega$. We need the following lemma:

Lemma 4.10. *Let $\{h\}$ be a $\Pi^0_1(z)$ singleton in $C_b(\omega_{\rm cof})$. If h is z-bounded-to-one, then h is z'-computable.*

Proof. Fix a z-computable function b such that $h^{-1}\{n\} \subseteq [0, b(n)]$ for any $n \in \omega$. Let p be a z-computable name of $\{h\}$ as a closed set in $C_b(\omega_{\rm cof})$. That is, p specifies a z-computable sequence $(\mathcal{G}(D_i))_{i\in\omega}$ such that $\{h\} = \bigcap_i \mathcal{G}(D_i)$. By b-boundedness, note that $h \in \mathcal{G}(D_i)$ iff

$$(\exists (A, a) \in D_i)(\exists i \in [0, b(a)] \setminus A)\; h(i) = a.$$

As each D_i is finite, $\mathcal{G}(D_i)$ determines an ω^ω-clopen set. Hence, $P = \bigcap_{i\in\omega} \mathcal{G}(D_i)$ is a $\Pi^0_1(z)$ set in ω^ω. Since $P = \{h\}$ is computably bounded, h must have a z-computable ω^ω-name. From this name, it is easy to get a z'-computable $C(\omega_{\rm cof})$-name of h. $\square$

Proof of Theorem 4.9. We show that for a $\Pi^0_1(z)$ singleton $\{h\}$, if h is not z''-computable, and $Y \leq_T \{h\}$ for $Y \in \mathbb{S}^\omega$, then Y is computable (or equivalently, Y is c.e. as a subset of ω).

Suppose that $Y \leq_T \{h\}$ for some $Y \in \mathbb{S}^\omega$. Let Ψ witness that $Y \leq_T \{h\}$, that is, for any name p of $\{h\}$ as a closed set in $C(\omega_{\rm cof})$, $\Psi(p)$ enumerates all elements of A. Recall that each name q of a closed set in $C(\omega_{\rm cof})$ is a sequence $(D_i)_{i\in\omega}$ of finite sets such that each $(A, a) \in D_i$ represents $\{g : g^{-1}\{a\} \not\subseteq A\}$. Then we define $D^\ell_{i,s} = D_i \cup \{([0, s], \ell)\}$, and $q^\ell(\langle i, s\rangle) = D^\ell_{i,s}$. As in the proof of Lemma 4.5, one can see that q^ℓ determines the same closed set as q.

First consider the case that there is $\ell \in \omega$ such that for all q, $\Psi(q^\ell)$ only enumerates a subset of Y. In this case, since $\Psi(p^\ell)$ enumerates all elements of A for any name p of $\{h\}$, by enumerating all computations of the form $\Psi(q^\ell)$, we obtain a computable enumeration of Y, that is, Y is computable.

Then, we now assume that for every $\ell \in \omega$ there is q such that $\Psi(q^\ell \upharpoonright t)$ for some t enumerates $m \notin Y$. Note that such $q^\ell \upharpoonright t$ does not extend to a name of $\{h\}$. This means that there is $\langle i, s\rangle < t$ such that for any $(A, a) \in D^\ell_{i,s}$ we have

$h^{-1}\{a\} \subseteq A$. In particular, $h^{-1}\{\ell\} \subseteq [0,s]$ since $([0,s],\ell) \in D^{\ell}_{i,s}$. As $s < t$, we also have $h^{-1}\{\ell\} \subseteq [0,t]$. Using Y', given ℓ one can compute such a t. Hence, h is Y'-bounded-to-one.

Since $\{h\}$ is a $\Pi^0_1(z)$ singleton and $Y \leq_T \{h\}$, Y is z-computable; hence Y' is z'-computable. By Lemma 4.10, this implies that h is z''-computable. This contradicts our choice of h. □

We can generalize the above result to show the following:

Theorem 4.11. *There are points in $(\omega^\omega)^{\mathsf{d}}$ and $(C_b(\omega_{\mathrm{cof}}))^{\mathsf{d}}$ which are $\mathcal{O}(\mathbb{Q})$-quasiminimal.*

Proof. Suppose that $\{x\}^{\mathsf{c}} = \omega^\omega \setminus \{x\} \in \mathcal{O}(\omega^\omega)$ computes $U \in \mathcal{O}(\mathbb{Q})$ via Ψ. First consider the case that there is $\ell \in \omega$ such that for all p, if $p(n)$ is longer than ℓ for all n, then $\Psi(p)$ only enumerates a subset of U. In this case, since there is an $\mathcal{O}(\omega^\omega)$-name of x consisting only of strings longer than ℓ, by enumerating all computations of the form $\Psi(p \upharpoonright u)$ such that the length of $p(n)$ is greater than ℓ for all $n < u$, we obtain a computable name of U.

We now consider the second case: For every $\ell \in \omega$ there is p such that $p(n)$ is longer than ℓ for all n and $\Psi(p)$ enumerates some $r \notin U$. Note that for such p there must be n such that $p(n)$ is of the form $x \upharpoonright k$ for some $k > \ell$ since, if $\Psi(p)$ enumerates an element $r \notin U$, then there must be n such that $p(n) \prec x$. Moreover, if $\Psi(p)$ enumerates r, then there is $u \in \omega$ such that $\Psi(p \upharpoonright u)$ enumerates a rational open interval including r. Let $(E_s)_{s\in\omega}$ be an enumeration of all finite sets $\{p(0), \dots, p(u-1)\}$ of finite strings such that $\Psi(p \upharpoonright u)$ enumerates an interval including some $r \notin U$. Note that for an effective enumeration (q_n) of all rationals, the set $\{n : q_n \in \mathbb{Q} \setminus U\}$ is computable in the jump of any name of U, and so is E_s.

Then, for any $s \in \omega$ there is $\sigma \in E_s$ with $\sigma \prec x$. Define $m_s = \min\{|\sigma| : \sigma \in E_s\}$. By our assumption, $(m_s)_{s\in\omega}$ is unbounded. If $\max_{t<s} m_t < m_s$ then define $g_s(n) = \max\{\sigma(n) : \sigma \in E_s\}$ for any $\max_{t<s} m_t \leq n < m_s$. This gives a function g dominating x, and such a g is computable in the jump of any name of U. As g dominates x, one can compute the ω^ω-name of x from g and any $\mathcal{O}(\omega^\omega)$-name of $\{x\}^{\mathsf{c}}$.

We now show that if $\{x\}$ is a $\Pi^0_1(z)$ singleton (i.e., z is an $\mathcal{O}(\omega^\omega)$-name of $\{x\}^{\mathsf{c}}$) and $x \not\leq_T z'$ then the second case never happens. Note that such an x exists for any given z. As $U \leq_T \{x\}^{\mathsf{c}}$, any name p of $\{x\}^{\mathsf{c}}$ computes a name of U. Choose a z-computable name of $\{x\}^{\mathsf{c}}$, which also computes a name of U. If the second case of the above argument happens, then z' computes a sequence (E_s) and therefore a function g dominating x. Hence, by the above argument, z' computes x, which contradicts our choice of x. Consequently, we always proceeds the first case, and this implies that U is computable.

For $\mathcal{O}C_b(\omega_{\mathsf{cof}})$, combine this argument and the proof of Theorem 4.9. □

5. Complexity issues

5.1. **Complexity of network.** Recall that a topological space X has an admissible representation iff X is T_0 and has a countable cs-network (and has a countable k-network whenever X is sequential) [39, 37]. We are now interested in the topological complexity of such a network. If X is not second-countable, such a network cannot be open. On the other hand, if such an X is regular, then it has a countable closed cs-network. Moreover, it is known that the Kleene-Kreisel spaces have a countable closed cs-network, where a Γ network is a network all of whose elements are in Γ.

Proposition 5.1. *For any $k \in \omega$, $\omega\langle k\rangle$ has a countable closed cs-network.*

Proof. By Normann [33, Lemma 3.13]. □

In general, even if a space has a countable cs-network, it does not necessarily have a Borel cs-network.

Proposition 5.2. *$C(\omega^\omega, \omega_{\rm cof})$ does not have a countable Borel cs-network.*

Proof. Recall that $[\sigma, k] = \{G \in C(\omega^\omega) : k \notin G[\sigma]\}$ yields a (standard) countable cs-network $\mathcal{N}_{\rm st}$ for $C(\omega^\omega, \omega_{\rm cof})$. Let $\mathcal{N}$ be an arbitrary cs-network for $X = C(\omega^\omega, \omega_{\rm cof})$. Let S be the collection of all subsets of $C(\omega^\omega, \omega_{\rm cof})$ which can be written as an intersection of finite sets $\{[\sigma_i, k_i]\}_{i<\ell}$ of basic sets. Then consider a subnetwork $\mathcal{M} = \{N \in \mathcal{N} : (\exists A \in S)\ A \subseteq N\}$.

Claim. $\mathcal{M}$ forms a cs-network.

Proof. To see this, we again consider the admissible representation δ induced from the standard cs-network $\mathcal{N}_{\rm st}$ (not from $\mathcal{N}$) for X. Since $\delta \colon \subseteq \omega^\omega \to X$ is continuous, as in the proof of [38, Theorem 12], for any $p \in {\rm dom}(\delta)$ and any open neighborhood U of $\delta(p)$, one can see that there are $N \in \mathcal{N}$ and n such that $\delta[p \upharpoonright n] \subseteq N \subseteq U$. Note that $\delta[p \upharpoonright n]$ is an intersection of finite sets $\{[\sigma_i, k_i]\}_{i<\ell}$ of basic sets. Hence, $N \in \mathcal{M}$. □

By the above claim, an enumeration $(N_i)_{i\in\omega}$ of $\mathcal{M}$ induces an admissible representation δ' of X. Since $\delta' \colon \subseteq \omega^\omega \to X$ is continuous, again by the proof of [38, Theorem 12], for any $p \in {\rm dom}(\delta)$ and any open neighborhood U of $\delta(p)$, there is σ, k such that $\delta'[p \upharpoonright n] \subseteq [\sigma, k] \subseteq U$. In particular, we have $N \in \mathcal{M}$ such that $\bigcap_{i<\ell}[\tau_i, j_i] \subseteq N \subseteq [\sigma, k]$.

Claim. N is not Borel.

Proof. Put $m = \max\{|\tau_i|, |\sigma| : i < \ell\}$. Let $A \subseteq \omega^\omega$ be a Π^1_1 complete set. Then, there is R such that $x \in A$ iff for all $h \in \omega^\omega$ there is n such that $(h \upharpoonright n, x \upharpoonright n) \in R$. If n does not bound v such that $(h \upharpoonright v, x \upharpoonright v) \in R$, then we remove the n-th element of $\omega \setminus \{k\}$ from $\Phi_x(\rho * h \upharpoonright n)$ for any $\rho \in \omega^m$. If n is the least number such that $(h \upharpoonright n, x \upharpoonright n) \in R$, then choose a large number $t > \max\{j_i, k, n : i < \ell\}$, and remove all elements of $\omega \setminus \{t\}$ from $\Phi_x(\rho * h \upharpoonright n)$ for any $\rho \in \omega^m$. Note that Φ_x defines G_x such that if $x \notin A$ then $G_x(h) = k$ for all h, and if $x \in A$ then there is h such that $G_x(\rho * h) \notin \{j_i, k : i < \ell\}$ for all $\rho \in \omega^m$. This implies that, if $x \notin A$ then $G_x \notin [\sigma, k] \supseteq N$, and if $x \in A$ then $G_x \in \bigcap_{i<\ell}[\tau_i, j_i] \subseteq N$. Therefore, $x \in A$ if and only if $G_x \in N$. We always have $\Phi(x) := \Phi_x \in {\rm dom}(\delta)$, and that $x \in A$ if and only if $\Phi_x \in (\delta')^{-1}[N]$.

Suppose that $(\delta')^{-1}[N] = B \cap {\rm dom}(\delta')$. Since the range of Φ is contained in ${\rm dom}(\delta')$, we have $\Phi^{-1}[(\delta')^{-1}[N]] = \Phi^{-1}[B] = A$. Hence, if B is Borel, so is A, which contradicts Π^1_1-completeness of A. Consequently, there is no such Borel set B, and thus N is not Borel. □

Consequently, every countable cs-network for $C(\omega^\omega, \omega_{\rm cof})$ contains a non-Borel set. In other words, $C(\omega^\omega, \omega_{\rm cof})$ has no countable Borel cs-network. □

One can extend this result to the second level of the projective hierarchy:

Proposition 5.3. *$C(\omega\langle 2\rangle, \omega_{\rm cof})$ does not have a countable $\utilde{\mathbf{\Delta}}^1_2$ cs-network.*

We need a lemma to prove this. Let δ_k is the standard admissible representation of $\omega\langle k\rangle$. We say that $\eta\in\omega^{<\omega}$ is $(n+1)$-*nontrivial* if $\delta_{n+1}[\eta]$ contains more than one elements. In other words, $\eta=(\tau_i,k_i)_{i<\ell}$ is $(n+1)$-nontrivial iff τ_i is not an empty string, and $k_i=k_j$ whenever $\delta_n[\tau_i]\cap\delta_n[\tau_j]$ is nonempty.

For $\psi\in\omega\langle 2\rangle$, let h_ψ be the *trace of* ψ, that is, $h_\psi(n)=\psi(\sigma_n*0^\omega)$, where $(\sigma_n)_{n\in\omega}$ is an enumeration of finite strings.

Lemma 5.4. *Let $B\subseteq\omega^\omega$ be a Π^1_2 set, and η be a 2-nontrivial string. Then, there is a primitive recursive predicate R such that*

$$\begin{aligned}\alpha\in B &\iff (\forall\psi\in\omega\langle 2\rangle)(\exists n)\ R(\alpha\upharpoonright n,h_\psi\upharpoonright n)\\ &\iff (\forall\psi\in\delta_2[\eta])(\exists n)\ R(\alpha\upharpoonright n,h_\psi\upharpoonright n).\end{aligned}$$

Proof. Let $\eta=(\tau_i,k_i)_{i<\ell}$ be a given 2-nontrivial string. By 2-nontriviality of η, there are pairwise incomparable strings $\lambda,\lambda'\in\bigcup_{i<\ell}[\tau_i]$. Let $A\subseteq\omega^\omega$ be a Π^1_1 set such that $\alpha\notin B\iff(\exists\beta)\ \langle\alpha,\beta\rangle\in A$. Then there is a primitive recursive predicate S such that

$$(\exists\beta)\ \langle\alpha,\beta\rangle\in A\iff(\exists\beta\in\omega^\omega)(\forall x\in\omega^\omega)(\exists n)\ S(\alpha\upharpoonright n,\beta\upharpoonright n,x\upharpoonright n).$$

Note that if the latter condition holds, one can define $\psi(\lambda*x)=x\upharpoonright n$ for the least n such that $S(\alpha\upharpoonright n,\beta\upharpoonright n,x\upharpoonright n)$, $\psi(\lambda'*i*x)=\beta(i)$, and $\psi(\tau_i*x)=k_i$ for any $x\in\omega^\omega$, and $\psi(y)=0$ for other y. Clearly, $\psi\in\delta_2[\eta]$.

By considering this ψ, one can construct a primitive recursive predicate R ensuring the following equivalences:

$$\begin{aligned}(\exists\beta)\ \langle\alpha,\beta\rangle\in A &\iff (\exists\psi\in\omega\langle 2\rangle)(\forall n)\ R(\alpha\upharpoonright n,h_\psi\upharpoonright n)\\ &\iff (\exists\psi\in\delta_2[\eta])(\forall n)\ R(\alpha\upharpoonright n,h_\psi\upharpoonright n).\end{aligned}$$

More explicitly, let s_σ and t_i be indices such that $h_\psi(s_\sigma)=\psi(\lambda\sigma 0^\omega)$ and $h_\psi(t_i)=\psi(\lambda' i0^\omega)$. Then, define $R(u,v)$ by $(\exists\sigma<|v|)\ S(u,v(t_i)_{i<|v(s_\sigma)|},v(s_\sigma))$. Clearly, R is primitive recursive, and it is not hard to check that this predicate has the desired property. This completes the proof. $\square$

Proof of Proposition 5.3. Let δ_2 be an admissible representation of $\omega\langle 2\rangle$. Recall that

$$[\sigma,m]=\{G\in C(\omega\langle 2\rangle,\omega_{\rm cof}):(\forall x\in\delta_2[\sigma])\ m\notin G(x)\}$$

yields a (standard) countable cs-network $\mathcal{N}_{\rm st}$ for $C(\omega\langle 2\rangle,\omega_{\rm cof})$. Let $\mathcal{N}$ be an arbitrary cs-network for $X=C(\omega\langle 2\rangle,\omega_{\rm cof})$. As in the proof of Proposition 5.2, we have $N\in\mathcal{N}$ such that $\bigcap_{i<\ell}[\tau_i,j_i]\subseteq N\subseteq[\sigma,m]$. Moreover, one can assume that σ is 2-nontrivial by choosing U in Proposition 5.2 as a sufficiently small open set.

Claim. N is not $\utilde{\boldsymbol{\Delta}}^1_2$.

Proof. Let $B\subseteq\omega^\omega$ be a Π^1_2 complete set. As σ is 2-nontrivial, choose a primitive recursive predicate R as in Lemma 5.4.

Now, note that given a name of $\psi\in\omega\langle 2\rangle$, one can effectively compute the total information of the trace h_ψ, that is, one can compute the sequence $(h_\psi(n))_{n\in\omega}$ in this order. Let $\alpha\in\omega^\omega$ and a name p of $\psi\in\omega\langle 2\rangle$ be given. If n does not bound k such that $(\alpha\upharpoonright k,h_\psi\upharpoonright k)\in R$, then we remove the n-th element of $\omega\setminus\{k\}$ from $\Phi_x(p)$. If n is the least number such that $(\alpha\upharpoonright n,h_\psi\upharpoonright n)\in R$, then choose a large number $t>\max\{j_i,k,n:i<\ell\}$, and remove all elements of $\omega\setminus\{t\}$ from $\Phi_x(p)$. As such an n is independent of the choice of a name p of ψ, the function Φ_x induces a function $G_x : \omega\langle 2\rangle \to \omega_{\rm cof}$. By our choice of R, if $x \in A$ then $G_x(\psi) \notin \{j_i, k : i < \ell\}$ for all ψ, and if $x \notin A$ then there is $\psi \in \delta_2[\sigma]$ such that $G_x(\psi) = k$. This implies that, if $x \in A$ then $G_x \in \bigcap_{i<\ell}[\tau_i, j_i] \subseteq N$, and if $x \notin A$ then $G_x \notin [\sigma, k] \supseteq N$. Therefore, $x \in A$ if and only if $G_x \in N$. We always have $\Phi(x) := \Phi_x \in \mathrm{dom}(\delta)$, and that $x \in A$ if and only if $\Phi_x \in (\delta')^{-1}[N]$.

Suppose that $(\delta')^{-1}[N] = B \cap \mathrm{dom}(\delta')$. Since the range of Φ is contained in $\mathrm{dom}(\delta')$, we have $\Phi^{-1}[(\delta')^{-1}[N]] = \Phi^{-1}[B] = A$. Hence, if B is $\utilde{\mathbf{\Delta}}^1_2$, so is A, which contradicts Π^1_2-completeness of A. Consequently, there is no such $\utilde{\mathbf{\Delta}}^1_2$ set B, and thus N is not $\utilde{\mathbf{\Delta}}^1_2$. □

Consequently, every countable cs-network for $C(\omega^\omega, \omega_{\rm cof})$ contains a non-$\utilde{\mathbf{\Delta}}^1_2$ set. In other words, $C(\omega^\omega, \omega_{\rm cof})$ has no countable $\utilde{\mathbf{\Delta}}^1_2$ cs-network. □

A similar argument also shows the following:

Proposition 5.5. *The hyperspace $\mathcal{O}(\omega^\omega)$ has no countable Borel cs-network, and the hyperspace $\mathcal{O}(\omega\langle 2\rangle)$ has no countable $\utilde{\mathbf{\Delta}}^1_2$ cs-network.* □

For a space X represented by δ, a set $S \subseteq X$ is Γ*-realized* if the set of all names of points in S is Γ in the set of all names of points in X (w.r.t. the Baire topology on the names); that is, $\delta^{-1}[S]$ is Γ in $\mathrm{dom}(\delta)$. If Γ is a topological pointclass, then every Γ set is Γ-realized; however the converse is not necessarily true. For instance, Hoyrup [14] showed that there is a Σ^0_2-realized subset of $\mathcal{O}(\omega^\omega)$ which is not Borel.

Proposition 5.6. *Every $\mathsf{CB}_0\langle k\rangle$-space has a countable $\utilde{\mathbf{\Pi}}^1_k$-realized cs-network.*

Proof. Let $C(X,Y)$ be a $\mathsf{CB}\langle k+1\rangle$ space. Then one can assume that X is a $\mathsf{CB}\langle k\rangle$ space, and Y is a $\mathsf{CB}\langle 0\rangle$ space, i.e., a T_0-space with a countable basis $(B_e)_{e\in\omega}$. By induction hypothesis, let $(N_d)_{d\in\omega}$ be a countable $\utilde{\mathbf{\Pi}}^1_k$-realized cs-network for X. Then define $M_{D,e}$ be the set of all f such that if $x \in N_d$ for any $d \in D$ then $f(x) \in B_e$. We claim that (M_{D_e}) is a countable $\utilde{\mathbf{\Pi}}^1_{k+1}$-realized cs-network for $C(X,Y)$. A name p of a point in $C(X,Y)$ is a point in $M_{D,e}$ if and only if for any name q of a point in $\bigcap_{d\in D} N_d$, an initial segment of p accepts an initial segment of q and enumerates e. By induction hypothesis, being a name of a point in $\bigcap_{d\in D} N_d$ is a $\utilde{\mathbf{\Pi}}^1_k$ property. Therefore, it is easy to check that the above condition is a $\utilde{\mathbf{\Pi}}^1_{k+1}$ property. □

Note that our proofs actually show the realized versions of Propositions 5.2, 5.3, and 5.5. For instance, the hyperspace $\mathcal{O}(\omega^\omega)$ has no countable Borel-realized cs-network. On the other hand, by Proposition 5.6, $\mathcal{O}(\omega^\omega)$ has a countable Π^1_1-realized cs-network.

5.2. **Distance from total degrees.** The argument in the beginning of Section 5 suggests that every $\omega\langle k\rangle$-degree is *arithmetically* equivalent to a total degree. One can also see that every point x in a CB_0-space is arithmetically equivalent to the jump of x (which is total). Here, a reasonable definition of a jump is one that uses an enumeration $(U^X_e)_{e\in\omega}$ of all effectively open sets in the space X. That is, for a point x in X, the jump $x'(e)$ is given by the truth value of $x \in U^X_e$; see e.g. [12]. However, Hoyrup [14] showed that, for any $k \geq 2$, the computable open subsets of $\omega\langle k\rangle$ are not Σ^1_k-enumerable. Here, we say that $S \subseteq X$ is Γ*-enumerable* if S is a computable image of some Γ subset of ω. Hence, we do not have the appropriate notion of the jump for $\mathsf{CB}_0\langle k\rangle$-space for $k > 0$.

We first see that if a space X has a countable hyperarithmetical cs-network, then every point $x \in X$ is hyperarithmetically equivalent to a total degree (which is a degree of the principal associate of x).

Observation 5.7. *Let X be a space having a countable $\Delta^0_{1+\alpha}$ cs-network, and consider the induced representation of X. Then for any $x \in X$ there is $z \in 2^\omega$ (i.e., a point of total degree) such that one can effectively find a pair of computable functionals Φ, Ψ witnessing the equivalence $p^{(\alpha)} \equiv_T z$ for any name p of x.*

Proof. Let $\mathcal{N} = (N_e)_{e\in\omega}$ be a countable $\Delta^0_{1+\alpha}$ cs-network of X, and let δ be the induced representation. Note that $\delta^{-1}[N_e]$ is also $\Delta^0_{1+\alpha}$ in $\mathrm{dom}(\delta)$. Therefore, for any name p of x, the set $z = \{e \in \omega : p \in \delta^{-1}[N_e]\}$ is computable in $p^{(\alpha)}$ uniformly in p. Consider z as a point in 2^ω. Then z has a total degree. Moreover, $e \in z$ if and only if $x = \delta(p) \in N_e$, and therefore, z is independent of the choice of p, and any enumeration of z is a name of x. □

It is clear that Observation 5.7 can be extended to any Δ^1_n levels. For instance, if a space X has a countable Δ^1_2 cs-network, then every point $x \in X$ is Δ^1_2-equivalent to a total degree.

We have seen in Proposition 5.5 that $\mathcal{O}(\omega^\omega)$ has no countable Borel cs-network. Therefore, Observation 5.7 is not applicable for $\mathcal{O}(\omega^\omega)$. Indeed, one can show that there is no uniform procedure witnessing that every $U \in \mathcal{O}(\omega^\omega)$ is hyperarithmetically equivalent to a point of total degree.

Proposition 5.8. *There is no hyperarithmetical functionals Φ and Ψ such that, for any $U \in \mathcal{O}(\omega^\omega)$, there is a point z of total degree such that Φ and Ψ witness that p is hyperarithmetically equivalent to z for any name p of x.*

Proof. Suppose not. Consider the whole space $U = \omega^\omega$, and then $\Psi(U) = z$ and $\Phi(z) = U$. In particular, Ψ is injective. Let ψ be a realizer of Φ. Note that any $p \in \omega^\omega$ is a name of an open set U_p, and therefore ψ is total. Then, p is a name of U (that is, for all $x \in \omega^\omega$ there is n such that p accepts $x \upharpoonright n$; which is a Π^1_1-complete property) if and only if $\Psi(U_p) = z$. Since ψ is Δ^1_1-measurable, and $\{z\}$ is closed in 2^ω, the equality $\Psi(U_p) = z$ is a Δ^1_1 property. This contradicts Π^1_1-completeness of the property that p is a name of U. □

5.3. **Enumerability of open sets.** Hoyrup [14] showed that the computable open subsets of $\omega^\omega \times \mathcal{O}(\omega^\omega)$ and $C(\omega^\omega, 2)$ are neither Σ^1_1-enumerable nor enumerable relative to any Δ^1_1 oracle. A similar technique is applicable for showing the following:

Theorem 5.9. *None of the following are Σ^1_1-enumerable or enumerable relative to any Δ^1_1 oracle:*

(1) *The computable open subsets of $C(\mathbb{Q})$.*
(2) *The computable open subsets of $\omega^\omega \times C(\omega^\omega, \omega_{\mathrm{cof}})$.*
(3) *The computable open subsets of $\omega^\omega \times \omega_{\mathrm{cof}}\langle 2\rangle$.*

To show Theorem 5.9, we use the notion of a fixed-point free multifunction. A multi-valued function (or simply a multifunction) $h\colon X \rightrightarrows X$ is *fixed-point free* if there is no $y \in h(y)$; see [14, 15]. Hoyrup [14] showed that $\mathcal{O}\mathcal{O}\omega^\omega$ and $\mathcal{O}C(\omega^\omega, 2)$ admits a computable fixed-point free multifuncion. It is easy to extend Hoyrup's result as follows:

Lemma 5.10. *Let Y be a space which is nontrivial in the sense that Y has a nonempty c.e. open set $V \subsetneq Y$. Moreover, assume that $C(\omega^\omega, Y)$ has a computable separable representation. Then, $\mathcal{OC}(\omega^\omega, Y)$ admits a computable fixed-point free multifunction.*

Proof. The proof is almost identical to Hoyrup [14]. Consider a nonempty c.e. open set $V \subsetneq Y$, and fix $z \in Y \setminus V$. Given a name p of $\mathcal{U}_p \in \mathcal{OC}(\omega^\omega, Y)$, we construct $\alpha \in \omega^\omega$ such that $\mathcal{H}_p := \{h \in C(\omega^\omega, Y) : h(\alpha) \in V\} \neq \mathcal{U}_p$.

By brute-force, one can look for a finite string which is accepted by $\mathcal{U}_p$. Until seeing such a finite string we guess that $\alpha = 0^\omega$. Assume that we found such a finite string at stage s. By computable separability, one can extend such a string to a name of a point h in $C(\omega^\omega, Y)$. Then $\mathcal{U}_p$ accepts such a name, i.e., $h \in \mathcal{U}_p$.

Now consider a slow name q of h such that the value of h on $[0^s]$ is not determined at finite stages. (Such a q exists since $[0^s]$ can be partitioned into infinitely many basic clopen sets.) The name q is also accepted by $\mathcal{U}_p$ at some finite stage. Then, one can find a largest clopen set $C \subsetneq [0^s]$ such that the value of h on C is determined by q up to this stage. One can effectively choose $\alpha \in [0^s] \setminus C$.

Define $\tilde{h}(x) = h(x)$ if $x \in [0^s]^{\mathsf{c}} \cup C$, and $\tilde{h}(x) = z$ if $x \in [0^s] \setminus C$. Note that $\tilde{h}$ is continuous: For any open set U, it is not hard to see that $\tilde{h}^{-1}[U] = h^{-1}[U] \cup [0^s] \setminus C$ if $z \in U$; otherwise $\tilde{h}^{-1}[U] = (h^{-1}[U] \setminus [0^s]) \cup (h^{-1}[U] \cap C)$ if $z \notin U$. Thus, $\tilde{h}^{-1}[U]$ is open since $h^{-1}[U]$ is open, and $[0^s]$ and C are clopen.

It is easy to see that we also have $\tilde{h} \in \mathcal{U}_p$, and $\tilde{h}(\alpha) = z \notin V$. Therefore, we have $\tilde{h} \notin \mathcal{H}_p$, which implies that $\mathcal{H}_p \neq \mathcal{U}_p$ as desired. □

Remark. The above proof only uses the property that every clopen set in ω^ω is (effectively) partitioned into infinitely many clopen sets. Hence, by Observation 2.28, one can replace ω^ω with $\mathbb{Q}$ in the above proof.

Corollary 5.11. *$\mathcal{OC}(\omega^\omega, \omega_{\rm cof})$ admits a computable fixed-point free multifunction. Similarly, $\mathcal{OC}(\mathbb{Q})$ admits a computable fixed-point free multifunction.*

A similar idea is applicable to show the existence of a computable fixed-point free multifunction in $\mathcal{O}(\omega_{\rm cof}\langle 2\rangle)$.

Lemma 5.12. *The hyperspace $\mathcal{O}(\omega_{\rm cof}\langle 2\rangle)$ admits a computable fixed-point free multifunction.*

Proof. Given a name p of $\mathcal{U}_p \in \mathcal{O}(\omega_{\rm cof}\langle 2\rangle)$, we construct $\alpha \in \omega_{\rm cof}\langle 1\rangle$ such that $\mathcal{H}_p := \{h \in \omega_{\rm cof}\langle 1\rangle : h(\alpha) \neq 0\} \neq \mathcal{U}_p$.

By brute-force, one can look for a finite string σ which is accepted by $\mathcal{U}_p$. Until seeing such a finite string, our candidate for $\alpha \in \omega_{\rm cof}\langle 1\rangle$ is 0^ω, which is named by declaring $\alpha^{-1}\{1\} \subseteq \emptyset$, $\alpha^{-1}\{2\} \subseteq \emptyset$, $\alpha^{-1}\{3\} \subseteq \emptyset$, $\dots$. Assume that we found such a finite string σ at stage s. Then we have already declared $\alpha^{-1}\{1\} \subseteq \emptyset, \alpha^{-1}\{2\} \subseteq \emptyset, \dots, \alpha^{-1}\{s\} \subseteq \emptyset$. Note that $\omega_{\rm cof}\langle 2\rangle$ is computably separable since any partial name is extended to a name of a constant function. Thus, one can extend σ to a name of a constant function c in $\omega_{\rm cof}\langle 2\rangle$. Then, $\mathcal{U}_p$ accepts such a name, i.e., $c \in \mathcal{U}_p$.

Assume that $c(x) = c_j(x) = j$ for any $x \in \omega_{\rm cof}\langle 1\rangle$. Then, consider the sequence $q_0 = (\{\langle [0,t], s+1\rangle\}, k)_{t\in\omega, k\neq j}$, each of whose entry declares that if $h^{-1}\{s+1\} \subseteq [0,t]$ then $H(h) \neq k$. We also consider $q_1 = (\{\langle \emptyset, u\rangle : s+1 \neq u \leq s+2\}, k)_{k\neq j}$, each of whose entry declares that if $h^{-1}\{u\} = \emptyset$ whenever $s+1 \neq u \leq s+2$ then

$H(h) \neq k$. Let H be a function which is consistent with the declaration made by the sequence $q := q_0 \oplus q_1$.

Note that if h is not constant, then h is finite-to-one. Therefore, $h^{-1}\{s+1\} \subseteq [0,t]$ for some t. Thus, by the declaration of q_0, we have $H(h) \neq k$ for any $k \neq j$. However, the sequence q never declare $H(h) \neq j$; thus $H(h) = j$. If $h^{-1}\{s+1\} \not\subseteq [0,t]$ for some t, then h must be the constant function c_{s+1} defined by $c_{s+1}(x) = s+1$ for any x. In this case, $h^{-1}\{u\} = \emptyset$ for any $u \neq s+1$. Thus, by the declaration of q_1, we have $H(h) \neq k$ for any $k \neq j$. By the similar argument as above, we have $H(h) = j$. This argument concludes that q is a name of the constant function $c_j \in \omega_{\rm cof}\langle 2\rangle$.

As $c_j \in \mathcal{U}_p$, a finite initial segment τ of $p \oplus q$ is accepted by $\mathcal{U}_p$. Let v be a sufficiently large number which is not mentioned in such an initial segment. Then our current name of α can be extended to a name of α satisfying that $\alpha(n) = 0$ for any $n < v$, and $\alpha(n+v) = n+s+1$ for any $n \in \omega$ by declaring $\alpha^{-1}\{0\} \subseteq [0,v)$, $\alpha^{-1}\{n+s+1\} \subseteq \{n+v\}$ for any $n \in \omega$.

We extend τ to a name r_H of another function $H \in \omega_{\rm cof}\langle 2\rangle$. Recall that the partial name τ has already specified the following information:

$$\begin{aligned} &H^{-1}\{k\} \subseteq \{h \in \omega_{\rm cof}\langle 1\rangle : h^{-1}\{s+1\} \not\subseteq [0,t]\} && \text{for } j \neq k < v \text{ and } t < v, \\ &H^{-1}\{k\} \subseteq \bigcup_{\substack{u \leq s+2 \\ u \neq s+1}} \{h \in \omega_{\rm cof}\langle 1\rangle : h^{-1}\{u\} \neq \emptyset\} && \text{for } j \neq k < v. \end{aligned}$$

Note that any extension of $\alpha \upharpoonright v{+}2$ is included in the right-hand sets of the above equations since $\alpha(v) = s+1$ and $\alpha(v+1) = s+2$. Let $(\tau_i)_{i\in\omega}$ be a (repetition-free) list of all finite strings of length $v+2$, and assume that $\tau_0 = \alpha \upharpoonright v+2$. Recall from the observation before Theorem 3.18, the clopen set $[\tau_i]$ can be written as an intersection of sets of the form $\mathcal{G}(D)$. Hence, one can extend τ to a name r_H declaring that $H^{-1}\{0\} = [\alpha \upharpoonright v+2]$, $H^{-1}\{k\} = \emptyset$ for any $0 < k < v$, and $H^{-1}\{v+n\} = [\tau_{n+1}]$ for any n.

Since τ is already accepted by $\mathcal{U}_p$ and r_H extends τ, the name r_H is also accepted by $\mathcal{U}_p$, i.e., $H \in \mathcal{U}_p$. On the other hand, $H(\alpha) = 0$ implies that $H \notin \mathcal{H}_p$. Consequently, we get that $\mathcal{H}_p \neq \mathcal{U}_p$ as desired. □

Proof of Theorem 5.9. By Corollary 5.11, Lemma 5.12, and Hoyrup [14, Theorem 5.1]. □

6. Linear realizable reducibility

Andrews et al. [1] showed that an enumeration degree is graph-cototal (i.e., an $\omega^\omega_{\rm cof}$-degree) if and only if it contains a cototal set A that reduces to $\omega \setminus A$ via a *unique axiom reduction* Γ, that is, if $n \in A$ then there is a unique D such that $(n, D) \in \Gamma$ and $D \cap A = \emptyset$. Moreover, such D can be assumed to be a singleton.

For nonempty oracles, we will call such a reduction as a linear reduction. More precisely, for $A, B \subseteq \omega$, we say that *A is linearly reducible to B* ($A \leq_{\rm lin} B$) if there exists a uniform c.e. sequence $(V_n)_{n\in\omega}$ such that for any $n \in \omega$, if $n \in A$ then $B \cap V_n$ is a singleton; otherwise, $B \cap V_n = \emptyset$. In this case, we also write $A = V(B)$, where $V = (V_n)_{n\in\omega}$. We rephrase the above result by [1] in this terminology: An e-degree is graph-cototal if and only if it contains a set which is linearly reducible to its complement.

A linear reduction is exactly a partial function on $\mathbb{S}^\omega$ tracked by a linear function (in the context of linear logic) between suitable coherent spaces. The notion of a linearly realizable function between spaces represented by coherent spaces has been introduced by Matsumoto-Terui [27] and Matsumoto [26]. For instance, Matsumoto-Terui [27] showed that a total function on $\mathbb{R}$ is linearly realizable (w.r.t. the canonical representation of $\mathbb{R}$) if and only if it is uniformly continuous.

Formally speaking, a *coherent space* is the set X of all cliques (complete subgraphs) of a reflexive graph $(|X|, E)$. For coherent spaces X, Y, a map $f\colon X \to Y$ is *linear* if it is monotone w.r.t. $\subseteq$, and, whenever $b \in f(x)$, there is a unique $a \in x$ such that $b \in f(\{a\})$. We consider a trivial coherent space $\mathcal{O}$ whose underlying graph is a countable infinite complete graph, that is, $|\mathcal{O}| = \omega$ and $E = \omega^2$. Then, cliques are subsets of ω, and computable monotone maps on $\mathcal{O}$ are enumeration operators. We say that a partial map f on $\mathcal{O}$ is *computable linear* if it is an enumeration operator, and whenever $x \in \mathrm{dom}(f)$ and $b \in f(x)$, there is a unique $a \in x$ such that $b \in f(\{a\})$.

Observation 6.1. *$A \leq_{\rm lin} B$ iff there is a partial computable linear map $f\colon\subseteq \mathcal{O} \to \mathcal{O}$ such that $B \in \mathrm{dom}(f)$ and $f(B) = A$.*

An equivalence class w.r.t. $\leq_{\rm lin}$ is called a lin-degree. We say that a lin-degree $\mathbf{a}$ is *graph-cototal* if it contains the complement of the graph of a total function on ω.

Proposition 6.2. *A* lin-*degree* $\mathbf{a}$ *is graph-cototal if and only if it contains $A \in \mathbf{a}$ such that $A \leq_{\rm lin} \omega \setminus A$.*

Proof. The following is just a careful analysis of the argument in Andrews et al. [1]. For the forward direction, if G and $\overline{G}$ are the graph and the co-graph of a total function on ω, then $V_{(x,y)} = \{(x,z) : z \neq y\}$ gives a reduction witnessing $\overline{G} \leq_{\rm lin} \omega \setminus \overline{G} = G$. For the reverse direction, given $A \neq \emptyset$ with $A \leq_{\rm lin} \omega \setminus A$ via V, define $g(a) = 0$ if $a \notin A$, and $g(a) = d+1$ if $a \in A$ and $(\omega \setminus A) \cap V_a = \{d\}$. Note that $g(a) = d+1$ if and only if $d \in V_a$ and $d \notin A$ by the uniqueness condition of V. Let $\overline{G}$ be the co-graph of g. Clearly, $T_a = \{(a, 0)\}$ witnesses $A \leq_{\rm lin} \overline{G}$. Moreover, define $U_{(a,0)} = \{a\}$ and if $d \in V_a$ then $U_{(a,d+1)} = \{d\}$; otherwise $U_{(a,d+1)} = \{p\}$, where p is a fixed element in A. It is easy to see that U witnesses $\overline{G} \leq_{\rm lin} A$. $\square$

We use $L(X, Y)$ to denote the space of all linearly realizable functions from X to Y. We now consider the lin-degrees of various subspaces of $\mathbb{S}^\omega$. Note that $C(\omega, X)$ and $L(\omega, X)$ are the same as a represented space. Therefore, we use X^ω to denote either $C(\omega, X)$ or $L(\omega, X)$.

Define $X\langle\langle 0\rangle\rangle = X$ and $X\langle\langle n+1\rangle\rangle = L(X\langle\langle n\rangle\rangle, X)$. We will see that the behavior of $X\langle\langle n\rangle\rangle$ is quite different from $X\langle n\rangle$. For $X = \omega$, we identify $a \in \omega$ with $\{a\} \in \mathbb{S}^\omega$.

Proposition 6.3. *For any $n \in \omega$, the $\omega\langle\langle n\rangle\rangle$-*lin-*degrees are exactly the 2^ω-*lin-*degrees.*

Proof. For $n = 1$, it is easy to check that (the graph G of) $g \in \omega^\omega$ is lin-equivalent to (the graph of) the characteristic function of G.

For $n = 2$, let V be a name of $g \in \omega\langle\langle 2\rangle\rangle$. Then, $g(x) = n$ iff there is $(\ell, k) \in V_n$ such that $x(\ell) = k$. There is a unique ℓ satisfying the above condition for any (some) x, n. Otherwise, there are $(\ell, k) \in V_n$ and $(\ell', k') \in V_{n'}$ with $\ell \neq \ell'$. In this case, let x be such that $x(\ell) = k$ and $x(\ell') = k'$. Then this implies $g(x) = n = n'$, and it is witnessed by two axioms $(\ell,k),(\ell',k')\in V_n$, which contradicts with the uniqueness condition.

We claim that, if g is not constant, such an ℓ is independent of the choice of V. Suppose that there are two names V,V' of g with different witnesses $\ell\neq\ell'$. Let x,y such that $g(x)\neq g(y)$. Then we have $(\ell,x(\ell))\in V_{g(x)}$ and $(\ell',y(\ell'))\in V'_{g(y)}$. If $\ell\neq\ell'$ then there is $z\in\omega^\omega$ such that $z(\ell)=x(\ell)$ and $z(\ell')=y(\ell')$. However, this must imply that $g(z)=g(x)$ and $g(z)=g(y)$, a contradiction.

If g is constant, g is clearly computable, so we now assume that g is not constant. Note that the above claim actually shows that a name of g is unique. Fix ℓ as above. Then, define $g\in\omega^\omega$ by $\tilde{g}(k)=\langle\ell,n\rangle$. Let V be the unique name of g. Clearly, $\langle\ell,k\rangle\in V_n$ if and only if $\tilde{g}(k)=\langle\ell,n\rangle$. This implies that $g\equiv_{\mathrm{lin}}\tilde{g}$. Hence, g has a ω^ω-lin-degree. □

For $c\leq\omega$, we identify $x\in c^\omega_{\mathrm{cof}}$ with the complement of a graph of a total function $g_x\colon\omega\to c$. Although it does not represent "cofinite" anymore, we abuse notation by writing c_{cof}. It is clear that 2^ω is (computably) linearly isomorphic to 2^ω_{cof}. If $c<\omega$, then c^ω_{cof} is computably homeomorphic to 2^ω, and so c^ω_{cof} is not interesting at all in the context of T-degrees; however we see that c^ω_{cof} has nontrivial lin-degrees.

Proposition 6.4. *For any $c\leq\omega$, there is a $(c+1)^\omega_{\mathrm{cof}}$-*lin-*degree which is quasi-minimal w.r.t. c^ω_{cof}-*lin-*degrees.*

Proof. Let $g\in(c+1)^\omega$ be sufficiently generic w.r.t. the standard Baire topology on $(c+1)^\omega$. We show that g as a point in $(c+1)^\omega_{\mathrm{cof}}$ which is quasi-minimal w.r.t. c^ω_{cof}-lin-degrees. Let σ be an initial segment of g, and let $V=(V_i)$ be a uniform c.e. sequence. First consider the case that there is n such that for any $k<c$ there is $(m_k,a_k)\in V_{(n,k)}$ which is consistent with σ, that is, if $m_k\leq|\sigma|$ then $a_k=\sigma(m_k)$. In this case, there is a string τ extending σ such that $(m_k,\tau(m_k))\in V_{(n,k)}$. If $g\in(c+1)^\omega$ extends such τ then for the co-graph $\overline{G}$ of g, $(n,k)\in V(\overline{G})$ for any $k<c$. Therefore, $V(\overline{G})$ is not the co-graph of an element in c^ω. It remains to consider the case that for any n, there is $k<c$ such that if (m_k,a_k) is consistent with σ then $(m_k,a_k)\notin V_{(n,k)}$. Let $g\in(c+1)^\omega$ be an extension of σ, and $\overline{G}$ be the co-graph of g. If the above k is not unique for some n, then clearly $V(\overline{G})$ is not the co-graph of an element in c^ω. If the above k is unique for any n, then $n\mapsto k$ is computable by waiting for a stage s such that for any $j<c$ with $j\neq k$, some (m,a) consistent with σ in enumerated into $V_{(n,j)}$ by stage s. Note that $V(\overline{G})$ must be the co-graph of the computable function $n\mapsto k$, and therefore $V(\overline{G})$ is c.e. □

In particular, the collection of the graph-cototal-lin-degrees (i.e., $\omega^\omega_{\mathrm{cof}}$-lin-degrees) strictly contains 2^ω-lin-degrees. Hereafter, we identify $a\in\omega_{\mathrm{cof}}$ with $\omega\setminus\{a\}\in\mathbb{S}^\omega$.

Proposition 6.5. *$\omega_{\mathrm{cof}}\langle\!\langle 1\rangle\!\rangle$ is a* lin-*subspace of $\omega\langle\!\langle 1\rangle\!\rangle=\omega^\omega$ consisting exactly of all permutations on ω. Hence, the $\omega_{\mathrm{cof}}\langle\!\langle 1\rangle\!\rangle$-*lin-*degrees are exactly the 2^ω-*lin-*degrees.*

Proof. First, it is clear that every permutation on ω is linearly realizable as a function on ω_{cof}. Now, a linearly realizable function $g\colon\omega_{\mathrm{cof}}\to\omega_{\mathrm{cof}}$ is induced from a uniform c.e. sequence $(V_n)_{n\in\omega}$. In other words, $(V_n)_{n\in\omega}$ codes an $\omega_{\mathrm{cof}}\langle\!\langle 1\rangle\!\rangle$-name of g. Note that $g(a)\neq n$ iff V_n contains some $b\neq a$. In particular, $a\in V_n$ declares $g^{-1}\{n\}\subseteq\{a\}$, which implies that g is injective unless g is constant.

We show that g has to be surjective. If not, there is d such that $g(n)\neq d$ for any n. Then, V_d is nonempty, say $a\in V_d$. However, we also have $g(a)\neq d$, and therefore, V_d contains some $b \neq a$. We now have $\{a, b\} \subseteq V_d$. However, if $k \notin \{a, b\}$ then $g(k) \neq c$ is witnessed by two axioms $a \in V_d$ and $b \in V_d$, which contradicts with the uniqueness condition. Consequently, g is a permutation on ω.

It remains to show that the inclusion map $\omega_{\rm cof}\langle\!\langle 1\rangle\!\rangle \subseteq \omega\langle\!\langle 1\rangle\!\rangle$ is computably linearly realizable. We claim that $g(a) = n$ if and only if $a \in V_n$. For the "only if" direction, assume that $g(a) = n$. Since g is not constant, there is b such that $g(b) \neq n$. Therefore, we must have $c \in V_n$ for some $c \neq b$, which declares $g^{-1}\{n\} \subseteq \{c\}$. As $g(a) = n$, we have $a = c$, and thus, $a \in V_n$. For the "if" direction, assume that $a \in V_n$. Suppose that $g(a) \neq n$. Then we must have $b \in V_n$ for some $b \neq a$. Let $c \notin \{a, b\}$, and recall that $c \in \omega_{\rm cof}$ is identified with $\omega \setminus \{c\}$. Then $\{a, b\} \subseteq (\omega \setminus \{c\}) \cap V_n$, which contradicts with the uniqueness condition of $(V_n)_{n\in\omega}$. Hence, $g(a) = n$. Consequently, given a name (V_n) of g, one can compute the total information of g, and vice versa.

For the second assertion, let g be a total function on ω. Define $\tilde{g}(n, 0) = (n, g(n)+1)$, $\tilde{g}(n, g(n)+1) = (n, 0)$, and $\tilde{g}(n, j) = (n, j)$ for any $j \notin \{0, g(n)+1\}$. Clearly, $\tilde{g}$ is a permutation. It is easy to see that $g \leq_{\rm lin} \tilde{g}$ since $g(n) = k$ iff $\tilde{g}(n, 0) = (n, k + 1)$. To see that $\tilde{g} \leq_{\rm lin} g$, note that $\tilde{g}(n, 0) = (n, j + 1)$ and $\tilde{g}(n, j + 1) = 0$ if $g(n) = j$; if $g(n, j + 1) = j + 1$ if $g(n) = k$ for some $k \neq j$. Thus, $V_{n,0,n,j+1} = \{(n, j)\}$, $V_{n,j+1,n,0} = \{(n, j)\}$, and $V_{n,j+1,n,j+1} = \{(n, k) : k \neq j\}$ yield a reduction. Hence, every total function g is lin-equivalent to a permutation $\tilde{g}$. Consequently, every ω^ω-lin-degree is a $\omega_{\rm cof}\langle\!\langle 1\rangle\!\rangle$-lin-degree. □

Proposition 6.6. *The $\omega_{\rm cof}\langle\!\langle 2\rangle\!\rangle$-*lin*-degrees are exactly the graph-cototal-*lin*-degrees.*

Proof. For any $g \in \omega^\omega_{\rm cof}$, define $\hat{g}(x) = g(x(0))$. By Proposition 6.5, $\omega_{\rm cof}\langle\!\langle 1\rangle\!\rangle$ is a lin-subspace of $\omega\langle\!\langle 1\rangle\!\rangle = \omega^\omega$ consisting exactly of all permutations on ω. For any n, there is a permutation x on ω such that $x(0) = n$. Hence, it is easy to see that $\hat{g} \equiv_{\rm lin} g$. Therefore, every graph-cototal-lin-degree is an $\omega_{\rm cof}\langle\!\langle 2\rangle\!\rangle$-lin-degree.

Next, assume that $g \in \omega_{\rm cof}\langle\!\langle 2\rangle\!\rangle$, and let V be a name of g. If g is constant, then it is computable. Therefore, we assume that g is not constant. Note that $(a, b), (a', b') \in V_n$ implies $a = a'$. Otherwise, for a permutation x on ω such that $x(a) = b$ and $x(a') = b'$, the fact $g(x) \neq n$ is witnessed by two axioms $(a, b), (a', b') \in V_n$, which contradicts the uniqueness condition. Moreover, as g is not constant, V_n is nonempty. So, fix a unique a_n such that $(a_n, b) \in V_n$ for some b.

Note that there is a unique i such that if $(a_n, b) \notin V_n$ for some b then $a_n = a_i$. Suppose not. Then let $a_i \neq a_j$ be such that $(a_i, b_i) \notin V_i$ and $(a_j, b_j) \notin V_j$. For a permutation x on ω such that $x(a_i) = b_i$ and $x(a_j) = b_j$, our choice of a_i and a_j implies that $g(x) = i$ and $g(x) = j$, a contradiction.

Let i be as above. It is easy to see that such an i is independent of the choice of a name V of g. Given b, one can effectively construct a permutation x_b on ω such that $x_b(a_i) = b$. Define $\tilde{g}(b) = g(x_b)$. It is not hard to check that $\tilde{g} \equiv_{\rm lin} g$. Hence, every $\omega_{\rm cof}\langle\!\langle 2\rangle\!\rangle$-lin-degree is graph-cototal. □

7. Open questions

We have shown that a $C(\omega_{\rm cof})$-degree is not necessarily cototal.

Question 1. *Does every cototal degree have a $C(\omega_{\rm cof})$-degree?*

We have not analyzed the differences in the degree structures of the four function spaces $C(\omega_{\rm cof})$, $C(\omega^\omega_{\rm co}, \omega_{\rm cof})$, $C(\omega_{\rm cof}, \omega^\omega_{\rm co})$, and $C(\omega^\omega_{\rm co})$.

Question 2. *Does there exist a $C(\omega^\omega_{\rm co})$-degree which is neither a $C(\omega^\omega_{\rm co}, \omega_{\rm cof})$-degree nor a $C(\omega_{\rm cof}, \omega^\omega_{\rm co})$-degree?*

Question 3. *Does there exist a $C(\omega^\omega_{\rm co}, \omega_{\rm cof})$-degree which is not a $C(\omega_{\rm cof}, \omega^\omega_{\rm co})$-degree?*

Question 4. *Does there exist a $C(\omega_{\rm cof}, \omega^\omega_{\rm co})$-degree which is not a $C(\omega^\omega_{\rm co}, \omega_{\rm cof})$-degree?*

Question 5. *Does $\omega\langle 2\rangle$ contain a non-total e-degree?*

As $\omega^\omega_{\rm cof} \subseteq \omega_{\rm cof}\langle 1\rangle \subseteq \omega_{\rm cof}\langle 2\rangle$, we have an $\omega_{\rm cof}\langle 2\rangle$-degree which is not an $\omega\langle 2\rangle$-degree.

Question 6. *Does every $\omega\langle 2\rangle$-degree an $\omega_{\rm cof}\langle 2\rangle$-degree?*

We have shown the existence of an $\omega\langle 2\rangle$-degree which is quasimi-minimal w.r.t. all e-degrees, an $\omega\langle 2\rangle$-degree which is not an $\mathcal{O}(\mathbb{Q})$-degree, and an $\mathcal{O}(\omega^\omega)$-degree which is $\mathcal{O}(\mathbb{Q})$-quasiminimal.

Question 7. *Does there exist an $\omega\langle 2\rangle$-degree which is $\mathcal{O}(\mathbb{Q})$-quasiminimal?*

Question 8. *Does there exist an $\omega_{\rm cof}\langle 2\rangle$-degree which is quasiminimal w.r.t. all e-degrees?*

Question 9. *Does there exist an $\omega_{\rm cof}\langle k+1\rangle$-degree which is not an $\omega_{\rm cof}\langle k\rangle$-degree?*

To solve this question, one can first ask the following:

Question 10. *Is the set of names of a point in $\omega_{\rm cof}\langle 3\rangle$ Π^1_2-complete?*

The argument in the beginning of Section 5 suggests that every $\omega\langle k\rangle$-degree is *arithmetically* equivalent to a total degree. One can also see that every point x in a CB_0-space is arithmetically equivalent to the jump of x (which is total), but we do not have the notion of the jump for $\mathsf{CB}_0\langle k\rangle$-space for $k > 0$.

Question 11. *Does there exists a $\mathsf{CB}_0\langle 1\rangle$-degree which is not arithmetically equivalent to a $\mathsf{CB}_0\langle 0\rangle$-degree?*

Acknowledgements. Kihara was supported by JSPS KAKENHI (Grant Numbers 21H03392, 22K03401 and 23H03346). Kihara also would like to thank the Young Scholars Overseas Visit Program 2018 in Nagoya University for supporting his visit to Nanyang Technological University. Ng was supported by the Ministry of Education, Singapore, under its Academic Research Fund Tier 2 (MOE-T2EP20222-0018).

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